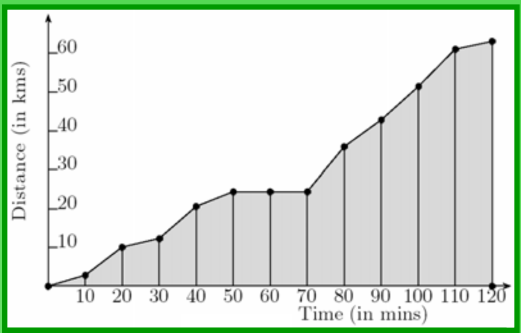
Basic Books in Science

Book 3

Relationships, Change and Mathematical Analysis



Roy McWeeny

Basic Books in Science

– a Series of books that start *at the beginning*

Book 3
Relationships, change, mathematical analysis

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Books in the Series will be available –*free of charge*–from the websites

<www.paricenter.com> (see 'Basic Books in Science')

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BASIC BOOKS IN SCIENCE

Acknowledgements

In a world increasingly driven by information technology and market forces, no educational experiment can expect to make a significant impact without the availability of effective bridges to the 'user community' – the students and their teachers.

In the case of "Basic Books in Science" (for brevity, "the Series"), these bridges have been provided as a result of the enthusiasm and good will of Dr. David Peat (The Pari Center for New Learning), who first offered to host the Series on his website, and of Dr. Jan Visser (The Learning Development Institute), who set up a parallel channel for further development of the project with the use of Distance Learning techniques. The credit for setting up and maintaining the bridgeheads, and for promoting the project in general, must go entirely to them.

Education is a global enterprise with no boundaries and, as such, is sure to meet linguistic difficulties: these will be ameliorated by the provision of translations into some of the world's more widely used languages. We are most grateful to Dr. Angel S. Sanz (Madrid), who has already prepared Spanish versions of the first few books in the Series: these are being posted on the websites indicated

as soon as they are ready. This represents a massive step forward: we are now seeking other translators, at first for French and Arabic editions.

The importance of having feedback from user groups, especially those in the Developing World, should not be underestimated. We are grateful for the interest shown by universities in Sub-Saharan Africa (e.g. University of the Western Cape and Kenyatta University), where trainee teachers are making use of the Series; and to the Illinois Mathematics and Science Academy (IMSA) where material from the Series is being used in teaching groups of refugee children from many parts of the world.

All who have contributed to the Series in any way are warmly thanked: they have given freely of their time and energy 'for the love of Science'. Paperback copies of the books in the Series will soon be available, but this will not jeopardize their free downloading from the Web.

Pisa 10 May 2007 Roy McWeeny (Series Editor)

BASIC BOOKS IN SCIENCE

About this Series

All human progress depends on **education**: to get it we need books and schools. Science Education is of key importance.

Unfortunately, books and schools are not always easy to find. But nowadays all the world's knowledge should be freely available to everyone – through the Internet that connects all the world's computers.

The aim of the Series is to bring basic knowledge in all areas of science within the reach of everyone. Every Book will cover in some depth a clearly defined area, starting from the very beginning and leading up to university level, and will be available on the Internet *at no cost to the reader*. To obtain a copy it should be enough to make a single visit to any library or public office with a personal computer and a telephone line. Each book will serve as one of the 'building blocks' out of which Science is built; and together they will form a 'give-away' science library.

About this book

This book, like the others in the Series, is written in simple English – the language most widely used in science and technology. It builds on the foundations laid in Book 1 (Number and symbols) and in Book 2 (Space) and deals with the mathematics we need in describing the *relationships* among the quantities we measure in Physics and the Physical Sciences in general. This leads us into the study of relationships and change, the starting point for Mathematical Analysis and the Calculus – which are needed in all branches of Science.

Notes to the Reader. When Chapters have several Sections they are numbered so that "Section 2.3" will mean "Chapter 2, Section 3". Similarly, "equation (2.3)" will mean "Chapter 2, equation 3". Important 'key' words are printed in **boldface**: they are collected in the Index at the end of the book, along with the numbers of the pages where you first find them.

Looking ahead –

The first two books in the Series introduced the 'language' of mathematics – how to *describe* the world around us by using symbols (marks on paper) to stand for the quantities we measure, such as distances and times. And, in Book 2, we used the concept of distance to build up all the main ideas of Euclid's geometry. But we didn't really study the ways in which different quantities might be *related* – how one quantity (y, say) may *depend on* another one (x).

A large part of mathematics is concerned with **relationships** and the description of such relationships in terms of symbols and equations. This field of study is called **mathematical analysis** and in the present Book you'll learn what it's all about and what you can do with it.

- Chapter 1 shows three ways of describing a relationship between two quantities: you can make a *table*, showing pairs of related values: or 'plot' the values in a *graph*, which gives you a 'picture' of the relationship; or (if you're lucky) you may find an **algebraic function** that gives the same related values. And in all cases you say "y is a function of x" and write "$y = f(x)$". There are many examples of simple functions in this chapter, what they are called, and how they look.

- At the heart of analysis is **the calculus**. The next three chapters introduce the essential ideas in its two main branches, the **differentiation** of a given function and its **integration**, starting from a very simple example – the distance s moved by a falling body as a function of the time (t) from its release: $s = f(t)$.

- Chapter 5 takes us back to the *representation* of any given function $y = f(x)$ as a **power series** $y = a_0 + a_1x + a_2x^2 + ...a_nx^n$.

- Finally, Chapter 6 takes a quick look at what you can do with all these 'tools': extending them to **functions of more than one variable**; solving **differential equations**, important throughout mathematics; and using their solutions to represent a given function in yet another way – as an **eigenfunction expansion** – fundamental in mathematical physics.

CONTENTS

Chapter 1

Relationships between quantities

1.1 Tables, graphs, functions

When we were talking about *measuring* physical quantities, at the beginning of Book 1, we spoke only of distance (measured with a measuring stick), time (measured with a clock), and mass (measured using a weighing machine). Mass, Length and Time (M,L,T) are three of the 'primary' quantities we meet in science; and in Book 2 we discovered how far it was possible to get by thinking about only two of them (L and T). So let's pick up from there. What might L and T stand for; and what do we mean by a **relationship** between the two?

Suppose s is the distance gone by a moving object (like a bicycle, or a truck, or a falling stone). It has dimensions [s]= L and is the length of the path, from the starting point (at time $t = t_0$, say) to the end point (at time $t = t_f$ – where f is short for 'final'). We'll use s and t to stand for the distance gone (s) and the time (t) when it gets there. The quantities s, t are **variables**, while s_0, t_0 will be *particular values* of the variables, at the beginning of the motion, and s_f, t_f will be the values at the end (final values). (Note that s_0, t_0, s_f, t_f are *not* variables but just certain values that the variables might have.) In Book 2 we nearly always measured distances along a straight line (an 'axis') and used x, for example, for a distance measured along the x-axis. But now we're talking about *any* kind of path, which could be 'snaky' like a serpent, so we'll call the variable s instead.

When we talk about a *relationship* between the variables s and t we simply mean the distance has a certain value (s) for any time (t) at which we measure it: s *depends on* the time t and we say s is the *dependent* variable, corresponding to the *independent* variable t. To give an example, the driver of a truck (which may be delivering sand to a building site) will have in front of him a clock and a 'counter' of some kind, to show how many kilometers he has driven (i.e. the corresponding value of s at time t). The relationship between the two can be decribed by giving a list of number-pairs, (s_0, t_0) at the start of the journey, (s_1, t_1) after the first 10 minutes, (s_2, t_2) after the second 10 minutes, and so on. The corresponding values can be set out in a

t	t_0	t_1	t_2	t_3	...	t_8	t_9	t_{10}	t_{11}	t_{12}
s	s_0	s_1	s_2	s_3	...	s_8	s_9	s_{10}	s_{11}	s_{12}

'Table' giving times (in 'units' of 10 minutes) and related distances (in km). Sometimes the pairs of values are listed automatically and the truck driver will get a printed list at the end of his journey. Let's suppose he gets the such a table of values: it gives an accurate record of his journey – a *tabular* way of showing the relationship between distance and time. But if he shows it to his boss it's not at first clear what he's been doing with his time!

t	0.0	10	20	30	40	50	60
s	0.0	2.8	10.1	12.3	20.6	29.4	29.4
t	70	80	90	100	110	120	
s	29.4	35.9	42.8	51.9	61.1	63.1	

Another way of showing the same relationship is much more *pictorial* and easy to understand: we make a 'picture', shown in Figure 1,

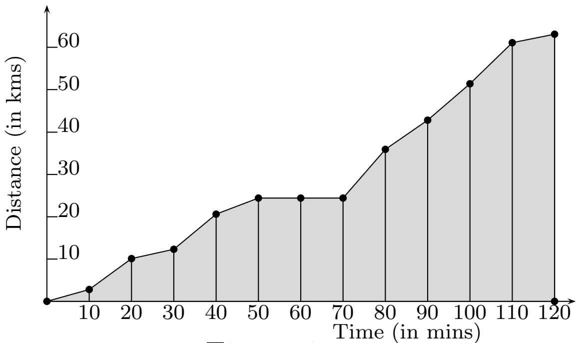


Figure 1

To make Fig.1, the pairs of values of t and s are taken as the x- and y-coordinates (Book 2, Section2.2) of a point in the picture and then all the points are connected together by short lines. This way you get a **graph** and it gives you a 'graphical representation' of the relationship between two variable quantities (t, s).

Of course we don't really know what happens *between* one point and the next, as the only values of s we have are for t-values 10 minutes apart – and a lot can happen in 10 minutes (there might be a breakdown and then the value of s won't change until things are put right). If instead we draw a *smooth curve* through all the points, then that will mean we suppose there are no sudden changes during any 10-minute interval – that things continue minute-by-minute in more or less the same way. Either way, Fig.1 gives only an *approximate* picture of the journey: but it's much easier to follow than a big table of numbers and we can see much more easily what it means. For example, the truck didn't go very far in the first 10 minutes (less than 3 km, while a good speed for a truck would be around 8 km in every 10 minutes); but the driver had to get the truck on the road and fill up with gasoline and so on. He did better in the second 10 minutes, going 7.3 km. But then in the next interval he only went 1.6 km! Why was that? In fact he had to stop at a quarry for a load of sand and gravel to take to the building site; and for most of the 10 minutes the truck wasn't moving at all. After that, he got onto a good road and tried to make up for the time lost. In the next 10 minutes he did 8.3 km and he followed that with 8.8 km – pretty good for an old truck! But he couldn't keep it up: he'd been on the way for nearly an hour, with some shovelling included, and was hot and tired and thirsty. If you look at the graph you'll see that the distance s didn't change at all during the next 20 minutes: he'd stopped for a drink and a chat with the other drivers – and the time passes quicker than you think!

In the first half of the second hour he got on well, except that he never did more than 7 km in any 10-minute interval; but then he did more than 9 km in each of the next two intervals. Why was that? In fact the road was going uphill for the first 14 km. Once he got to the top it was downhill for the rest of the way and he picked up speed, doing 18 km in the next 20 minutes. Then he came into the town, where there was traffic and bicycles and people crossing the road. He had to slow down and the last 2 km took him a full ten minutes.

Now you can see why the graphical description of the relationship between s and t is so useful: just a quick look at the graph, with no calculations of any kind, is enough to give you a good idea of what might be happening – and even why! We'll be using graphs throughout mathematics and physics; and through most of science.

There's a third way, however, of describing the relationship between two quantities x and y. It's sometimes possible to find a mathematical *rule*, a **formula**, for getting the value of y when you're given the value of x. To give an example, let's suppose you drop a small pebble from the roof of a very high building. The distance it falls in time t (we'll go on calling it s but don't get mixed up with 's' short for 'second' – the unit of time) depends on t according to the rule

$$s = ct^2, \tag{1.1}$$

where c is a constant, with the approximate value $c = 5\ \mathrm{ms}^{-2}$. Why do we write it in that strange way, instead of just saying $c = 5$? It's because physical quantities, the things we measure, are *not* 'just numbers' – they are numbers of *units*. Here m is the unit of length (L) – the metre – while s is the unit of time (T) – the 'second', and we say that c "has the dimensions" LT^{-2}. So if we use the formula (1.1) to find how far the pebble falls in two seconds we must put $t = 2$ s; and the answer will be

$$\begin{aligned} s &= c \times t \times t = 5 \text{ ms}^{-2} \times 2\text{s} \times 2\text{s} \\ &= (5 \times 2 \times 2)(\text{ms}^{-2}\text{s}^2) = 20 \text{ m}. \end{aligned}$$

The only arithmetic we need do is to multiply the numbers: the units 'look after themselves' and the result is correctly given as 20 metres. In general, a quantity with dimensions LT^{-2} will give us one with dimensions of length (L) when we multiply it by two time factors (T^2). It was already noted (in Book2, Section 2.1) that if we know the dimensions of a quantity it's easy to change our units of measurement: for if we take a new unit k times as big as the old unit then the numerical measure of the quantity will become k times *smaller*. Thus, if we decide to use the 'foot' instead of the metre as our unit of length (1 foot = 12 inches =12× 2.54 cm = 30.48 cm ≈ 0.30 m), then 20 m = (20/0.30) ft ≈ 66.67 ft; and the quantity c, with one factor L in its dimension formula, will become (in the same way) $c = (5/0.30)$ ft s^{-2} = 16.67 ft s^{-2}.

The third way of describing a relationship between two quantities, using a formula as in (1.1), is what we'll be talking about in most of this book. Whenever we can find a formula we can start to use the methods of *mathematical analysis*, even when the formula is much more complicated than (1.1) and even when we want to ask difficult questions about the relationship and what it means. But before going into the details let's just note that *any* relationship between two quantities – an independent variable x and a dependent variable y – can be written in a short form as

$$y = f(x), \tag{1.2}$$

which can be read as "y is a function of x". This just means that, for every value we may choose for x, there will be some related value of y – the variable that depends on x. We don't *need* to have a nice simple formula like (1.1): if we don't have one, then we must get the related values from a list that's been given to us, or read them off from a graph – which is just a more pictorial way of representing measured values. The 'function symbol' f simply means that if we're told x then we have a way of getting the related value of y.

Of course, a dependent variable may depend on several variable quantities, not just one. If, for example, we're climbing a hill we might go a distance x towards the East and then a distance y towards the North, arriving at a height z above the level we started from. And in that case we would write

$$z = f(x, y). \tag{1.3}$$

This equation will describe the surface of the hill: if you start from a point with coordinates (X, Y) and walk only in the East-West direction, keeping the value of y constant at $y = Y$, your height will be described by $z = f(x, Y)$ – a function of the single variable x; but if you go only in the North-South direction, with x having the constant value $x = X$, then your path will be described by the function $z = f(X, y)$ – with y as the only independent variable.

First, however, we'll be talking only about functions of a single variable, so we won't allow both x and y to vary at the same time; and we'll call the dependent variable y, as usual, not z.

1.2 Some simple functions – and what they look like

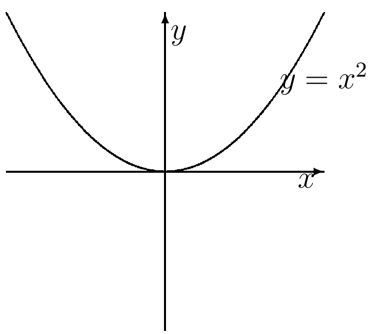


Figure 2

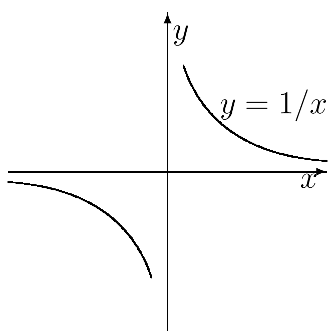


Figure 3

Often the relationship between two quantities can be represented by a simple formula such as (1.1). This is a special case of

$$y = f(x) = cx^n, \tag{1.4}$$

in which the independent and dependent variables have been called x and y and n has been given the value $n = 2$). We can *picture* the relationship by drawing a graph as in Section 1.1. Marking the points (x, y) for corresponding values of x and y and joining them by a smooth curve is called "plotting the curve".

The result is shown in Fig.2. This curve is called a **parabola**: it starts from $y = 0$ at $x = 0$ and rises smoothly to larger and larger values, without limit, as x increases. We say "y tends to infinity as x tends to infinity" or, in symbols, $y \to \infty$ as $x \to \infty$. When the independent variable is a time, $x = t$ as in (1.1), the 'branch' corresponding to *positive* values of t is the right-hand half of the parabola: the other half, on the left, corresponds to *negative* x values and the whole curve is *symmetric* across the y-axis, since the value of $y = x^2$ is unchanged if we use $-x$ instead of x. We also say the curve describes an *even* function of x, since it depends only on an even power ($n = 2$) of the independent variable.

If instead we take $n = -1$ we get the curve shown in Fig.3. The function $y = cx^{-1} = c/x$ describes a **hyper-**

bola: as x gets bigger and bigger y does the opposite. In symbols, $y \to 0$ as $x \to \infty$. But as $x \to 0$ something very special happens: $y \to \infty$ but in such a way that the curve becomes closer and closer to the vertical line $x = 0$ (i.e. the y-axis). The hyperbola has an **asymptote** at $x = 0$. On going to negative values of x, we see there is a **singularity** (a very special point) at $x = 0$; there the curve breaks into two separate branches, the part for x negative (but with $|x| > 0$) being just like the positive branch, but reflected across both the axes. Both the positive and negative branches also have horizontal asymptotes, where they come closer and closer to the x-axis. The whole curve represents an *odd* function of x, arising from the odd power ($n = -1$) of the independent variable.

The functions plotted in Figs.2 and 3 introduce two other important ideas. Away from any singular points there may be, they are both *continuous*: however close two x-values become, the related y-values also become indefinitely close – there are no 'breaks' or 'jumps' in continuous curves. Also, both functions are *single-valued*: if we are given the value of x then the function defines only *one* related value of the dependent variable y. Most of the time, in Science, we'll be talking about continuous and single-valued functions, but it's important to note that they may be defined only for a certain range of values of the independent variable: if those values lie only between $x = a$ and $x = b$ we say $f(x)$ is continuous and single-valued "in the interval (a, b)".

In Book 1, we met quite a number of important functions, which we come across again and again in all parts of Science: we need to know what to call them.

1.3 The naming of functions

There are two main kinds of function. Those described by a rule (a formula or equation) which involves only a *finite* number of 'elementary' operations like adding and multiplying (which includes raising to a power, x^n where n is an integer) are called **algebraic functions**. All other functional relationships, those which are *not* algebraic, are called **transcendental functions** – they go 'beyond' or 'above' those of algebraic form.

An example of the first kind is the relationship resulting from the equation $x^2 + 3xy + y^3 = 0$: it determines a value of y for any given values of x, even though it doesn't give a simple expression for $y = f(x)$, with y on one side of the = sign but not on the other. We say it gives "y as an an **implicit function** of x", whereas $y = f(x)$ gives y as an **explicit function** of x when the formula can actually be found.

The second kind of relationship (transcendental) includes all those which involve an *infinite* number of elementary operations, usually they arise from some kind of *limiting* process (Book 1, Chapter 4) in which the value of y is *approached* more and more closely, by taking more and more x-dependent terms. In Books 1 and 2 we already met functions like $\exp x$, $\sin x$, $\cos x$, which are of this kind. And again, in addition to those of *explicit* form, we may meet transcendental functions of *implicit* form, in which the equation defining the relationship contains a mixture of x-dependent and y-dependent terms.

Because implicitly defined functions are less common than the others and are harder to deal with in terms of what we know already, we'll mostly be talking about explicit algebraic functions; and sometimes explicit transcendental functions.

What kinds of algebraic function can we expect? An example helps: the relationships

$$(i)\ x^2 + 3xy - 4y = 0, \quad \text{and} \quad (ii)\ x^4 + 2x^2y^2 - 3y^2 = 0,$$

are both algebraic, but are implicit. Both equations can be solved to give the explicit forms:

$$(i)\ y = \frac{x^2}{4-3x}, \qquad (ii)\ y = \frac{x^2}{\sqrt{3-2x^2}}.$$

The first expresses y in terms of x using only a finite number of elementary operations (additions, multiplications, and their inverses – subtraction and division): it is called a **rational algebraic function**. Such functions can always be expressed in the form $y = f(x)/F(x)$, where $f(x)$ and $F(x)$ are both **polynomials** of the form $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$; it is a rational *integral* function. The second result (ii) is *not* of rational *integral* form because it involves the operation of taking a square root – and that is a *fractional* power. (It is possible to imagine algebraic functions that don't fit into any of these types but we'll have no use for them.)

As for the transcendental functions, the most important one we've already met is the **exponential function**. It is defined by an **infinite series**, see Section 5.1 of Book 1, and is usually denoted by $y = f(x) = \exp(x)$:

$$y = \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots , \qquad (1.5)$$

which is the sum of an *infinite* number of terms, each of the form $x^n/n!$ ($n!$, 'factorial n', being the product of the first n positive integers).

The behaviour of this function, also denoted by $y = e^x$, is shown in Fig.4. Its value increases smoothly as x goes from $-\infty$ to $+\infty$, the curve crossing the y-axis at the point where $x = 0, y = 1$.

Two other important examples are the 'sine' and 'cosine' functions (Chapter 4 of Book 1), defined by

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots, \quad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots. \tag{1.6}$$

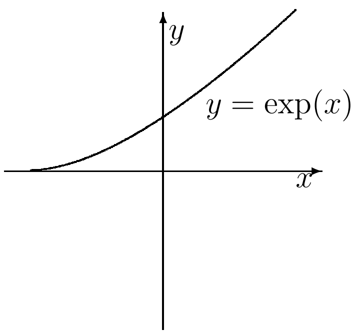


Figure 4

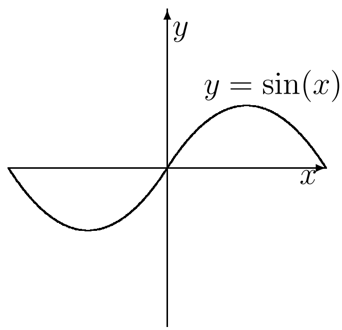


Figure 5

These functions relate the sine and cosine of an angle x, as defined in geometry (Book 2 Section 3.2), to the angle itself, expressed in *radians*. Both functions are **bounded**, y having an upper limit $+1$ and a lower limit -1, and are **periodic**, the same cycle of values repeating over and over again when x increases by 2π radians (360 degrees), The sine of x is shown in Fig.5: the cosine looks exactly the same but the curve is pushed through π radians along the x-axis.

1.4 Turning it round – inverse functions

The function $y = f(x)$, once it is written down as a formula, will give us the value of y that goes with any value we may choose for x: but what if we want to turn the question round and ask "What is the value of x that, when put in the formula, will give some chosen value of y? That's a very different question because the "independent variable" (the one we choose) has now been called y instead of x; and we may not have a formula that will give us the answer. Let's start with a case where we *do* know the answer: the function $y = f(x) = x^2$. The new function, which gives x in terms of y, can be written $x = g(y)$ to show that it's different from the one that works by squaring the variable. But in this case we already know the rule for getting x from a given value of y: the two formulas we need are

$$y = f(x) = x^2; \qquad x = g(y) = \sqrt{y}, \qquad (1.7)$$

where the second one is just the definition of what we mean by the "square root" of y – it's the number that, when squared, gives us the number x in the first formula (see Book 1, Section 4.2).

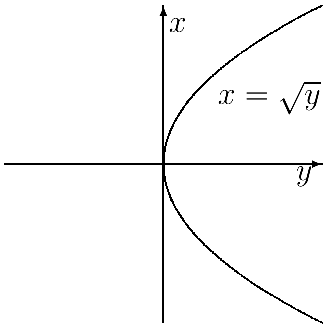


Figure 6

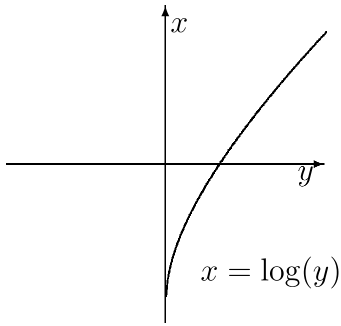


Figure 7

If we're using the graph to express y in terms of x, there's no problem; because any point on the curve tells us both (i) the value of y that goes with a given x, or (ii) the value of x that goes with a given y. But since we've always agreed that the independent variable is plotted along the horizontal axis, and the dependent variable along the vertical axis, we'll need to swap the two axes. The function $y = x^2$ shown in Fig.2 will then look like Fig.6, where values of y are plotted horizontally and those of x vertically.

Let's note, however, that the inverse of a function may have very different properties from the one we start from. One value of $y = x^2$ arises from *two* different values of the independent variable, x and $-x$. So $x = \pm\sqrt{y}$ is a two-valued function of y (taken as the positive square root).

The function $g(y) = \sqrt{y}$ can also be written as $g(x) = \sqrt{x}$ because it doesn't matter what name we give to the variable – it's often just called the **argument** of the function. In this case, $g(x)$ is called the **inverse** of the function $f(x) = x^2$. And similarly when $y = f(x) = x^n$, the inverse function $g(x)$ is called the "nth root" of x and is written $g(x) = \sqrt[n]{x}$.

In most cases, however, the formula for the inverse of a function is not known or can be hard to find. For example, the exponential function whose graph is shown in Fig.4 has an inverse function called the **logarithm** defined so that

$$\text{When} \quad y = f(x) = \exp x \quad \text{then} \quad x = g(y) = \log y. \tag{1.8}$$

And in this case, though it's easy enough to change the axes round in Fig.4 to get the function shown in Fig.7, it's much harder to find a *mathematical rule* for getting the inverse of the function defined by the series in (1.5): for that means finding a formula for x, on the right in equation (1.5), in terms of y on the left.

Note that with *named* functions, like $\exp(x), \sin(x)$, *etc.*, we usually leave out the brackets ('parentheses') round the variable x: instead of $\exp(x)$, for example, we just write $\exp x$; and we'll do this from now on. The round brackets are useful when the argument of the function is itself a big expression (e.g. $(3x^2 + 5)$ instead of just

x).

Many functions and their inverses arise when we try to solve problems that we meet in Science. For example, if the variable y (let's call it N) measures the number of male/female couples in a population of flies (or rabbits, or people) after n generations, the law of **exponential growth** has the general form

$$N = N_0 \exp(cn), \tag{1.9}$$

where N_0 is the number when you start counting and c is a constant which is bigger the faster they reproduce. The number n is a measure of time (the number of 'average life-spans' – which you might want to translate into years, for a human being, or days, for a fly). Fig.4 shows how fast the population grows (positive values of the argument cn). The population of the world is now roughly 10 thousand times what it was a thousand years ago. Of course, (1.9) gives the growth only if there are no restrictions: it applies for a simple 'model' in which we don't allow for death and disease. If we change the sign of the constant c in (1.9) the population N_0 will be *reduced* in every generation: the negative branch of the curve will then represent the law of **exponential decay**. In any real population the growth or decay will depend on many factors and one of the most important questions for humanity is how to make sure that the population of the world is *sustainable* – without depending on disease and death, starvation and wars, to hold it back.

Exercises

1) Plot the curve $y = x^2$ for values of x going from -5 to $+5$. Then do the same for $y = x^2+2$ and for $y = 2x^2+2$. If you take instead the function $y = px^2 + q$, p and q being numbers with any values you choose (i.e.'arbitrary' values), you get a curve something like the one in Fig.2, but try to describe how it will be changed.

2) Plot the curve $y = 1/x$ for x going from -1 to $+1$ in steps of 0.1. What happens when x is very close to zero? Are there any singularities in this range and, if so, for what values of x? Do the same for the function $y = 1/(x + \frac{1}{2})$.

3) What name would you give to the function $y = x^2/(4-3x)$? By putting in a few values of x and getting the corresponding values of y, make a rough drawing (a 'sketch') to show the shape of the curve. What happens (i) when x becomes very big and (ii) when it has values between 1.3 and 1.4 ? Are there any asymptotes and, if so, show them in your sketch.

4) Now look at the function $y = x^2/\sqrt{3 - 2x^2}$ and make a sketch to show how it behaves in the range from $x = 0$ to $x = \sqrt{3/2}$. What happens at the upper limit?

5) Calculate values of $y = \exp x$ for values $x_1 = 0.01$ and $x_2 = 0.02$, from the series (1.5), using only the first

four terms. Verify that the corresponding values of y are related by $y_1 \times y_2 \approx y_3$, where $y_3 = \exp(x_1 + x_2)$.

6) When $y = \exp x = e^x$, x is said to be the "*logarithm* of y to the base e". How can you describe the results in Exercise 5 in terms of logarithms?

7) Try to calculate the value of the fraction $y = f(x) = (x^2 - 4)/(x - 2)$ when $|x|$, the **modulus** of x (its value without the $\pm$ sign), is close to 2. Note that the result begins to look like 0/0, which is said to be *indeterminate*; but your values will suggest that this ratio gets closer and closer to 4 – we say it "tends to the limit 4". Show that by writing $(x^2 - 4) = (x + 2)(x - 2)$ you can prove that this value is the *exact* limit at $x = 2$. We write

$$lim_{x\to 2}\left[\frac{(x^2-4)}{(x-2)}\right] = 2 + x = 4.$$

8) The fraction in Exercise 7 approaches a limit also when x becomes as large as we please (indefinitely large, or 'infinite'). Show that the limiting value is x and is therefore also indefinitely large.

9) Show that the fraction $y = (2x^2 + 5)/(x^2 + 3x)$ approaches the *finite* limit $y = 2$ as $x \to \infty$. (The symbol ∞ stands for "infinity".) In this case then

$$lim_{x\to\infty}\left[\frac{(2x^2+5)}{(x^2+3x)}\right] = 2.$$

10) Use the first few terms in the series that define $y = \sin x$ and $y = \cos x$, given in (1.6) of this Chapter, to find the limits as $x \to 0$ of the following functions:

- (a) $y = (\sin x)/x$
- (b) $y = (1 - \cos x)/x^2$
- (c) $y = (\cos ax - 1)/x^2 \quad (a = \text{constant})$
- (d) $y = (x \cos x - x)/x^3$
- (e) $y = x \sin x + \cos x$

Chapter 2

Calculus – what's it all about?

2.1 Velocity of a falling body

In the last Chapter we were talking about the relationship between two variable quantities, one (the dependent variable) being determined by the value of the other (the independent variable). For example, the velocity v of a falling object depends on the time t (i.e. on how long it's been falling). We can use this simple example to introduce the main ideas of the branch of Mathematics usually called just 'the calculus', of which there are two kinds – the **differential calculus** and the **integral cal-**

culus. The first is about *rates of change*; if we use x, y to stand for the dependent and independent variable, how fast does y change when we make small changes in x? The second is about how the small changes in y, made one after another, can be added up – or "integrated" – to get the *total* change in y resulting from a big change in x.

The relationship between s, the distance gone by a falling pebble (starting from rest), and t, the time taken, is 'parabolic' as in Fig.2; but only the positive branch of the curve is needed (time *after* the start being counted positive). The curve is continuous, and s goes on increasing faster and faster as t increases (i.e. as time passes). As an equation,

$$s = s(t) = ct^2 \tag{2.1}$$

When we say "faster and faster" this brings in the idea of **velocity** (or 'speed' if you need only its magnitude). If the speed increases by an amount a in the first second, and by the same amount in the next second, and so on, we say the pebble is moving with 'constant **acceleration** a' and that v is *proportional* to t: if you double t then v will also be doubled. As an equation,

$$v = v(t) \propto t = at. \tag{2.2}$$

The two relationships (distance/time and velocity/time) are shown in Figs. 8 and 9.

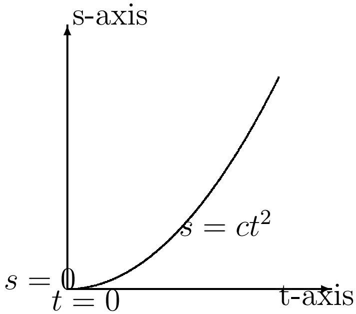


Figure 8

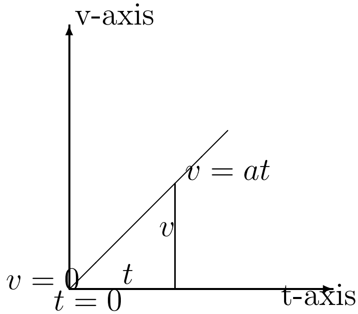


Figure 9

The 'proportionality constant' a, in (2.2) has the value

$$a = \text{constant} \approx 9.81 \text{ m s}^{-2}. \tag{2.3}$$

(Note that if you put $t = 1$ s in (2.2) you find: velocity after 1 s $= (9.81 \text{ m s}^{-2}) \times (1 \text{ s}) = 9.81 \text{ m s}^{-1}$ – so the units are all OK.)

The last two equations say nearly all we need to know about falling bodies and we'll be using them a lot. The important thing about a is that its value is roughly the same for dropping things of *any* kind, *anywhere* on the surface of the Earth! More about all this in Book 4 – here we we only want to study the mathematics.

Now look at the graph for $v = v(t)$, Fig.9, which is a straight line passing through the origin: it shows v as a *linear* function of t. The *slope* of the line is $a = v/t$, the increase in velocity (plotted in the 'up' direction) divided by the increase in time (plotted in the 'left-right' direction). It's just as if you were going up a hill: the slope measures the steepness of the hill. (We used slopes quite a lot in geometry, in Book 2.)

We now have three formulas: (i) $s = ct^2$, (ii) $v = at$, (iii) $a =$ constant. How are they connected?

What happens as the time t increases by a little bit δt? (Remember that the Greek letter δ isn't something *multiplying* t – it's short for "a little bit of".) So the time goes from t to $t + \delta t$, and v goes from v to $v + \delta v$ as in Fig.10:

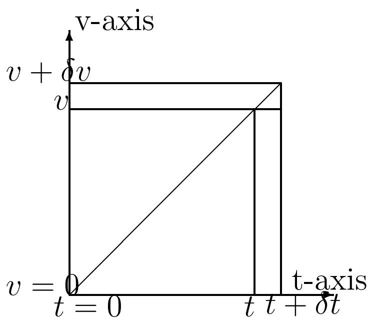


Figure 10

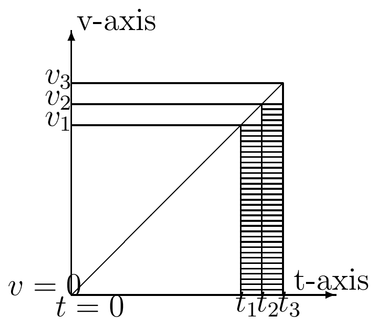


Figure 11

The constant a, which we've called the "acceleration", is the *rate of change of* v*, with respect to* t; it is the *slope* (v/t) of the straight line in Fig.10.

The distance fallen s also increases a bit; because something moving with velocity v will go a distance $v \times \delta t$ during the small time interval δt. Here we'll use the value

$v = v(t)$ at the beginning of the interval because it's not going to change much in such a small time change. To summarize, then:

> When the time increases from t to $t + \delta t$,
>
> v goes to $v + \delta v$, where $\delta v = a\delta t$ (a being the *slope* of the v-curve)
>
> s goes to $s + \delta s$, where $\delta s \approx v\delta t$ - and so δs, the *extra* distance gone during the interval δt, is roughly the *area* of a small rectangle of height v and width δt.

Now draw vertical lines up from the t-axis to meet the curve $v = v(t)$ in neighbouring points, like $(t_1, v_1), (t_2, v_2), (t_3, v_3)$ in Fig.11. Along with the increases in velocity

$$\delta v_1 = a\delta t_1, \quad \delta v_2 = a\delta t_2, \quad \delta v_3 = a\delta t_3,$$

there are corresponding increases in distance gone; and these are approximated by

$$\delta s_1 = v_1\delta t_1, \quad \delta s_2 = v_2\delta t_2, ...$$

which are represented by the areas of the shaded rectangles in Fig.11.

You can see what's coming. We divide the whole time from $t = 0$ up to $t = T$, say, when the pebble hits the ground, into small intervals δt: and then the *total* distance fallen in time T will be represented by the *sum* of the areas of all the strips, like those in Fig.11, between the limits $t = 0$ and $t = T$. As you can see in Fig.12, this is very nearly the same as the area of a triangle that holds all the strips; and this area is just *half* the area of the rectangular box whose horizontal and vertical sides have lengths T and V.

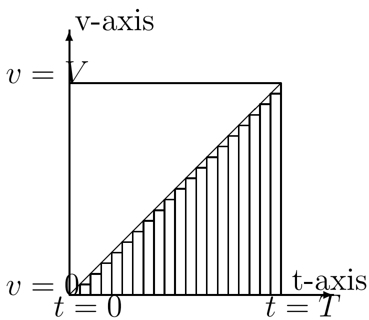


Figure 12

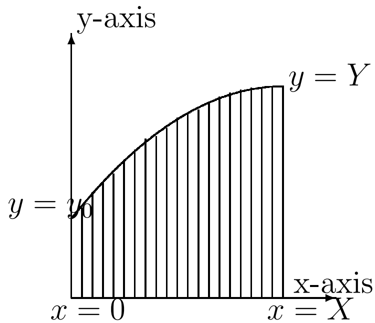


Figure 13

Since $V = aT$ we can get a formula for the total distance fallen in time T:

$$\text{Total distance gone} = \tfrac{1}{2}(T \times V) = \tfrac{1}{2}aT^2. \qquad (2.4)$$

This result shows us why the constant in (2.1) is different from that in (2.2): in fact, $c = \frac{1}{2}a$.

Two things have to be said. First, the falling stone is only an example and what we've done can be done just as well for *any* relationship $y = f(x)$ between two variable quantities – even when the slope of the curve (a) is not constant, giving you a real curve as in Fig.13 instead of a straight line. And, second, the results are approximate only because we considered small changes δt – but not *very, very small*, or "infinitesimal" changes: if we make the steps small enough the results can be made as accurate as we please, becoming *exact* 'in the limit as $\delta t \to 0$'. In the next Section we'll analyse these ideas of large and small, and limits, more carefully.

But before starting on 'mathematical analysis' let's summarize what we've done using only graphs and pictorial ideas. For any function $y = F(x)$, there is a rate of change of the dependent variable y, with respect to a change δx in the independent variable x; and this is the slope of the curve at the point (x, y) we're looking at – not exactly at the *point*, but in the small interval as the independent variable goes from x to $x + \delta x$.

In the example, the function we're now calling F gave the distance s in terms of the time t; and the rate of change of s with respect to t, the slope of the F-curve, was called the velocity. If we use the same word for the 'velocity of change', we can write $v = f(x)$, noting that v is no longer a *linear* function (as it was in the case of constant acceleration) but depends on which part of the curve we look at. So in general we're interested in two functions of the independent variable x, namely

$$y = F(x), \qquad v = f(x) = (\text{slope of } F(x) \text{ at point } x) \tag{2.5}$$

And in the last two Figures we've seen how the *area* under a curve such as $v = f(x)$, between two limits $x = 0$ and $x = X$ (using X to mean the upper limit of x), could be used to find the function $y = F(x)$. This is a marvellous result –because it means that, just as we can get the function $f(x)$ as the *slope* of the function $F(x)$, we can get $F(x)$ from a well-defined *area* under the other function $f(x)$. In other words, we have found a way of passing from one function to the other *in either direction*: finding the slope of a function is called 'differentiating' and leads to the **differential calculus**, while finding the area is called 'integrating' (putting together all the strips in Figs.12 or 13 to get the whole) and leads to the **integral calculus**. Very often the two branches of the calculus are studied separately; but they're really just 'different sides of the coin', integration being simply the **inverse** of differentiation. We look at them both together in the next two Sections.

2.2 Infinitely large and infinitely small – limits

There are three main branches of Mathematics. The first started with ideas about number and the use of symbols in arithmetic and algebra: it was developed quite a lot in Book 1. The second deals with ideas about space and geometry; it was taken quite a long way in Book 2. Both these branches had their origins in ancient times, so by now they are thousands of years old. But the third main branch, is much more recent; it began to develop only about three hundred years ago and is called Mathematical Analysis. It deals with many of the things we know something about already (e.g. the 'infinite series' we used in Book 1 to define numbers which couldn't be defined using only a finite number of 'elementary operations', such as multiplication and division); but it does so in a much more careful and precise way. In the last Section, we were thinking about 'infinite processes': for example, the rate of change of $y = f(x)$ was defined 'at a point' on the curve by looking at the *slope* of the curve in an interval whose end points became 'infinitely close together' – the infinite process being that of letting a small element of the curve shrink to a point; and another example was the division of an area into an indefinitely large number of infinitely narrow strips. These infinite processes are different from the ones we looked at in Book 1 (e.g. Chapter 5) because they deal with *functions*, not just with sequences of numbers. We also came across the idea of a *limit*, as the number to which the sum of a series of n terms can get as near as we please when n becomes larger and larger – without ever getting there! An example was the decimal number $0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + ... = 0.11111 ...$ representing the rational fraction 1/9; it gets closer and closer to the limit, but to get there you need the 'sum to infinity' where the series is never-ending.

The first kind of infinite process we're going to talk about in this Section is the calculation of the *slope* of a curve at the point P with coordinates (x, y).

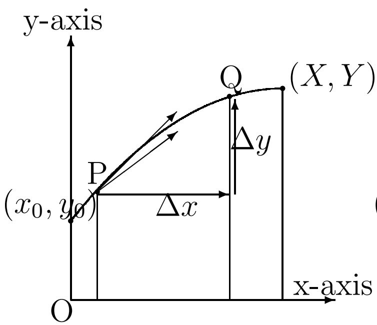


Figure 14

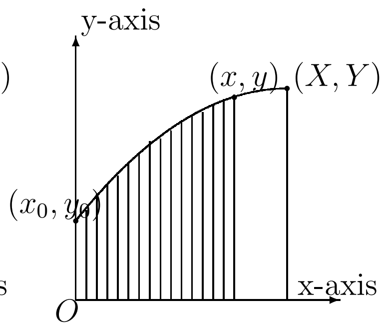


Figure 15

In Fig.14 we can define the *average* slope of the curve in the interval PQ as the fraction $\Delta y/\Delta x$ (distance 'up', divided by distance 'on'); but that's not the same as the slope "*at point* P". (Note that the end-points of the curve are shown as (x_0, y_0) and (X, Y), just as P is (x, y); while Q has coordinates $(x + \Delta x, y + \Delta y)$) Fig.14 gives us the idea of the rate of change of y as x increases – as the slope of a line – but that's all! In mathematical analysis we work with the *numerical ratios* $\Delta y/\Delta x$ and ask how they change as we look at smaller and smaller intervals. When Δx and Δy are finite (and in Fig.14 they are quite big) their ratio is only approximately the slope of the curve at the *point* P – it's what we've called the 'average' rate of change: what we really want is the slope of the line that just *touches* the curve at point P, which is called the **tangent** at P. But we're not going to get it by putting a ruler against the curve and making measurements: we're going to use only arithmetic.

We start by using very very small changes, calling them δx and δy instead of Δx and Δy, and get the ratio $\delta y/\delta x$ by actually doing the division. But we soon run into trouble as δx gets smaller and smaller. Take an example like $y = x^2$, at the point where $x = 6, y = 6 \times 6 = 36$. First make a small change $\delta x = 0.01$ and calculate the value of y at the new point where the x-value is $x+\delta x = 6.01$. The new y-value will be $y+\delta y = (6.01)^2 = 36.1201$, so the increase δy will be 0.1201 and the ratio $\delta y/\delta x$ will be $0.1201/0.01 = 12.01$.

Next let's take $\delta x = 0.001$, so as to get a value of $\delta y/\delta x$

very close to the point $x = 6, y = 36$. The result will be

$$\frac{\delta y}{\delta x} = \frac{36.012001 - 36.000000}{0.001} = \frac{0.012001}{0.001} = 12.001,$$

which seems to be almost exactly 12. If we go on long enough we'll find values closer and closer to 12; but to get them we have to calculate the function $y = x^2$ more and more accurately (we're already going to 6 figures after the decimal point) and then divide a δy that's almost 0 by a δx that's also close to zero. We're looking for a limit, in this case 12, which would seem to be close to 0/0 and this is nonsense. There must be a better way of getting it!

Another infinite process we met in the last Section is the evaluation of an area under a curve by dividing it into narrow strips: Fig.15 will remind you of the problem. The whole area we want to find, let's call it A, is bounded by the curve, the x-axis and the verticals at x_0 and x. In the Figure it is shown divided into strips; and if the strip of height y has a width δx (we're talking about *any* strip) then it has an area of approximately $\delta A = y \times \delta x$ – not exactly, because the top side is curved. As we've already noted, the approximation gets better and better as we use thinner strips and take many more of them: we'll get the correct limit *exactly* only if we take an infinite number of infinitely thin strips – but we don't yet know how to do it!

What we *can* say is that, if we add one more strip to the last one on the right, the area A will increase by δA; so we know the *rate of increase of the area* as the upper boundary moves – as $x \to x + \delta x$. It is the ratio $\delta A/\delta x$, and for the curve $y = x^2$ this becomes

$$\frac{\delta A}{\delta x} = \frac{y\delta x}{\delta x} = y = x^2$$

– which is an exact result, the δx in the numerator being cancelled exactly by the δx in the denominator.

To summarize: The area shown is a function of x, the position of the upper boundary, and in symbols

$$A = A(x), \quad \delta A/\delta x = f(x).$$

In other words: for any given function $y = f(x)$, we may not know the corresponding area-function $A(x)$, but we *do* know the *slope* of the curve $A = A(x)$ at any point (x, y).

In the next Section we come to the general question of how to calculate slopes and areas exactly – which is a central problem of the Calculus.

2.3 Derivatives, integrals – and how to find them

In the last Section we discovered that getting the slope of a curve at a given point, as the limit of a ratio $\delta y/\delta x$, was not so easy: as δx and δy both tend to zero it is not clear what their ratio will tend to, since 0 divided by 0 doesn't mean anything.

To get round this difficulty, let's use the same example $y = f(x) = x^2$ but simplify the fraction $\delta y/\delta x$ by expressing δy in terms of δx and seeing if anything will cancel. In this way we can get a result by using simple algebra, *before* we start doing any arithmetic.

When Δx and Δy in Fig.14 are replaced by the much smaller steps δx and δy, the y-coordinate of the upper point (Q) on the curve will become

$$y+\delta y = (x+\delta x)^2 = (x+\delta x)\times(x+\delta x) = x^2+2x\delta x+\delta x^2$$

and subtracting y $(= x^2)$ gives us the expression for δy:

$$\delta y = 2x\delta x + \delta x^2.$$

If we now divide both sides by δx we find the ratio

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

and this is true however small δx may become!

So there is a finite limit as $\delta x \to 0$ and it is $2x$. We write this result as

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 2x.$$

The limiting value of the ratio, which gives the slope of the curve in Fig.14 *at the point* P (i.e. as Q comes closer and closer), and is called the **derivative** at point $\mathrm{P}(x, y)$ of the function $y = f(x)$. It is usually written as $\mathrm{d}y/\mathrm{d}x$, to make it look like a ratio; but we must remember that it's just the name (we read it as "d-y-d-x") of a *single number* – the limiting value of the ratio $\delta y/\delta x$. (You may be wondering why the 'd' is written with a straight back – in 'Roman' rather than the usual 'italic' type, which is always used for numbers or quantities. It's again to make sure you don't think of $\mathrm{d}x$ as a product of d and x.)

In the last Section we found, for $x = 6$, the approximate result $\mathrm{d}y/\mathrm{d}x \approx 12.001$; but now we can say the *exact* value is $\mathrm{d}y/\mathrm{d}x = 2x = 12$.

We can also write down a formula for finding the derivative $\mathrm{d}y/\mathrm{d}x$ of *any* function $y = f(x)$ by going through the same steps – but without saying that the function is really just $y = x^2$. To get the numerator in the fraction $\delta y/\delta x$, before going to the limit for $\delta x \to 0$, we simply take the difference of two function values, value of $f(x + \delta x)$ (after the increase in x) minus value of $f(x)$

(before the increase); and then we divide by the increase δx itself. So

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

and then we go to the limit $x \to 0$ to get the single number $\mathrm{d}y/\mathrm{d}x$. So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \tag{2.6}$$

and that's how we **differentiate**, or "find the derivative of", any function whatever!

Now let's turn to the other infinite process we want to study – that of finding the area under a curve by dividing it into a large number of thin strips. To find the area we have to **integrate** the function $y = f(x)$, represented by the curve, and the result is called the *integral* of the function. More correctly, because the area depends on where we put the upper and lower boundaries (let's call them $x = X$ and $x = x_0$ as in Fig.15), it is the **definite integral** from $x = x_0$ to $x = X$.

To get the area we can start with strips all of width Δ, the first going from $x = 0$ to $x_1 = \Delta$, the next from $x_1 = \Delta$ to $x_2 = 2\Delta$, and so on until we get to the last at $X = N\Delta$. For a linear function $y = f(x) = x$, like that in Fig.11, the corresponding y values will be $y_1 = x_1,\ y_2 = x_2,\ ...\ y_N = X$; and the sum of the areas of all the N strips will be

$$S_N = y_1\Delta + y_2\Delta + \ ... \ y_N\Delta = (1 + 2 + 3 + \ ... \ N)\Delta^2$$

Now we know from Book 1 how to evaluate a series like this. We write it out again, in reverse order, so

$$S_N = (N + (N-1) + (N-2) + \ ... \ 1)\Delta^2$$

and add the two series together to get $2S_N = N \times (N + 1)$ (since there are N terms, each with the same value $(N + 1)$. The total area is thus

$$S_N = \tfrac{1}{2}N(N+1)\Delta^2 \qquad (2.7)$$

and we can see how it depends on the number of strips we've used.

What we need is the *limit* of (2.7) as N becomes infinitely large and the strips infinitely narrow. And we can get it in terms of the x-value marking the top boundary, because $X = N\Delta$ and therefore Δ depends on the number of strips: $\Delta = X/N$. If we put this result in (2.7) we find

$$S_N = \tfrac{1}{2}N(N+1)(X^2/N^2) = \tfrac{1}{2}X^2\left(1 + \tfrac{1}{N}\right). \qquad (2.8)$$

It's now clear that for $N \to \infty$, $S_N \to \frac{1}{2}X^2$. This limit of the sum is the *exact* area under the curve between the boundaries at $x = 0$ and $x = X$: it is the required *integral* of the function $y = f(x)$, from $x = 0$ to $x = X$ and we write

$$y = f(x) = x, \qquad \int_0^X f(x)\mathrm{d}x = \tfrac{1}{2}X^2. \qquad (2.9)$$

Of course, there's nothing special about the upper boundary $x = X$; or about the name that we use for the independent variable – we could just as well use t instead of x. So the integral in (2.9) can equally be written as

$$\int_0^x f(t)\mathrm{d}t.$$

When defined in this way it is known as the 'indefinite' integral of $f(x)$, usually written $\int f(x)\mathrm{d}x$. But more of this later!

The symbol $\int$ in (2.9) is a 'long S', a 'stretched out' S, meaning the limit of the 'Sum', and the dx is to remind us that we are integrating 'with respect to x' by dividing the whole area into strips of 'infinitesimal width'; the 0 and the X indicate the 'range' of the integration, from 0 at the bottom to X at the top. This is the notation nearly always used nowadays; but, as we noted at the end of Section 3.1, integration is really the *inverse* of differentiation – so we could use the symbol D, for example, for the *operation* of differentiating, and D^{-1} for the inverse operation of integrating. We'll say more about this in later Chapters, but the general idea is that if two functions $f(x)$ and $F(x)$ are related by

$$f(x) = \frac{\mathrm{d}F}{\mathrm{d}x}, \quad F(x) = \int f(x)\mathrm{d}x, \tag{2.10}$$

then we can just as well write

$$f(x) = \mathsf{D}F(x), \quad F(x) = \mathsf{D}^{-1}f(x). \tag{2.11}$$

While we're talking about notation, we should make a short list of things we're going to use later:

- Differentials In defining the derivative of a function $y = f(x)$, we said that $\mathrm{d}y/\mathrm{d}x$ was a single number (the limit of a ratio) and not the ratio of two different numbers, $\mathrm{d}y$ and $\mathrm{d}x$. But (as long as we're careful!) we *can* use the symbols $\mathrm{d}x$ and $\mathrm{d}y$ separately, calling them **differentials**.

  At the very beginning of this Section we used the example $y = f(x) = x^2$, finding that when x increased by an amount δx the corresponding change in y was $\delta y = 2x\delta x + \delta x^2$. The rate of increase of y with x, close to the point (x, y), was thus

  $$\frac{\delta y}{\delta x} = 2x + \delta x a$$

  and $\mathrm{d}y/\mathrm{d}x$ *at* the point (x, y) was defined as the limit of this ratio:

  $$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 2x.$$

If we now use $\mathrm{d}x$ as a new name for the small increase δx, calling it a "differential", it follows that

$$\delta y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x + \mathrm{d}x^2.$$

In other words, the actual increase in y will be $\delta y = (\mathrm{d}y/\mathrm{d}x)\mathrm{d}x + \mathrm{d}x^2$. This is why, in older books, the derivative $\mathrm{d}y/\mathrm{d}x$ is often called a "first differential coefficient" – meaning that, when δy is expressed in terms of powers of $\mathrm{d}x$, then $\mathrm{d}y/\mathrm{d}x$ is the first coefficient in the expansion.

The differential $\mathrm{d}y$ is defined simply by knocking off all terms in the expansion after the first. With this definition we can write

$$\mathrm{d}y = (\mathrm{d}y/\mathrm{d}x)\mathrm{d}x \tag{2.12}$$

and if we divide both sides of the equation by $\mathrm{d}x$ it follows that the derivative $\mathrm{d}y/\mathrm{d}x$ (single number) can be written as the ratio of two differentials, $\mathrm{d}y$ divided by $\mathrm{d}x$:

$$\mathrm{d}y \div \mathrm{d}x = (\mathrm{d}y/\mathrm{d}x) \tag{2.13}$$

- where the division symbol ($\div$) is used to show that the quantity on the left is really a ratio of two small quantities, rather than the differential coefficient $(\mathrm{d}y/\mathrm{d}x)$. But remember (2.13) follows from the way we defined dy; and that, however small we take dx, the differential dy will not be exactly equal to the corresponding change (δy) in y. We'll find many examples of how useful it can be to work with differentials.

- Repeated derivatives At the end of Section 3.1, the symbol D was used to mean the *operation* "differentiate with respect to x", so that

$$\mathsf{D}y = \mathsf{D}f(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$$

is just another notation for the derivative of a function. But since, in general, the derivative of $f(x)$ is another function of x (often denoted by $f'(x)$) we can differentiate a second time, obtaining

$$\mathsf{DD}y = \mathsf{DD}f(x) = \mathsf{D}f'(x) = f''(x)$$

– yet another function of x, which is called the **second derivative** of $f(x)$. And the process can be continued to obtain still higher derivatives.

If we use the original notation for differentiating, putting (d/dx) in place of D, this last equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}f'(x) = f''(x)$$

and this can be put in shorter form by writing

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) = f''(x). \tag{2.14}$$

This is the notation we usually use for second derivatives: it corresponds to taking the 'square' of the basic operation $\mathrm{d}/\mathrm{d}x$ and an 'nth derivative' can be denoted in a similar way by

$$\mathsf{D}^n y = \frac{\mathrm{d}^n y}{\mathrm{d}x^n}.$$

The second derivative is specially important: it means the "rate of increase of the *slope* of $f(x)$ at the point (x,y)" and if its value is zero then the curve of y against x has 'flattened out' at that point. The second derivative thus gives us a way of finding if a function has reached a **maximum** (biggest value), a **minimum** (smallest value), or at least a **turning point** (where the curve is neither at a maximum or a minimum, but is 'flat' or 'stationary'). This kind of information is very important in many applications of mathematical analysis, as we'll find later.

In the next Chapter we start making a collection of derivatives and integrals of some of the 'standard' functions we need most. But before doing that let's think about how we can use them: we can't make a collection big enough to include *all* the functions we might want to deal with – for there would be no end to it! On the other hand, if we know the derivatives and integrals for just a small list of functions then we can ask how our results can be *combined*.

2.4 Building up to bigger things

Suppose we have made a list of standard functions, $u = u(x),\ v = v(x), ...$ and their derivatives with respect to the variable x, $\mathrm{d}u/\mathrm{d}x = u'(x),\ \mathrm{d}v/\mathrm{d}x = v'(x), ...$. Notice that we're using the shorthand notation $u'(x)$ to mean the new function we get when we differentiate $u(x)$; and similarly for $v(x)$.

To build up to more complicated functions, we might start by simply *adding* two functions from the list, to get the new function $f(x) = u(x) + v(x)$; or we could multiply any function by a constant number c, taking $f(x) = cu(x)$; or we could multiply $u(x)$ and $v(x)$ by numbers a and b and *then* add them, getting $f(x) = au(x) + bv(x)$. And in each case we'll want to get the corresponding derivative $f'(x) = \mathrm{d}f/\mathrm{d}x$.

Another thing we might want to look at is the *product* of two function from our list, taking $f(x) = u(x)v(x)$ and again asking how we can get its derivative $f'(x)$.

Finally, we may want to study a function of a *new variable* $v = v(x)$, taking $f(x) = u(v)$ – which is certainly a function of x alone, since it is obtained by putting the variable v (in place of x) into the formula which defines $u(x)$.

We'll deal, in turn, with each of these three building methods: (i) differentiating a sum, (ii) differentiating a product, and (iii) differentiating a 'function of a function'. All three methods will be used over and over again in this book and many others, so it's important to learn how to use them.

(i) Differentiating a sum; $y = u + v$

Let's start from the definition of $\mathrm{d}y/\mathrm{d}x$ in equation (2.6):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}, \qquad (2.15)$$

which is good for *any* function $y = f(x)$. The derivative of $u(x)$ follows on replacing y by u in (2.15) and is therefore

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x} = \lim_{\delta x \to 0} \frac{u(x + \delta x) - u(x)}{\delta x}, \qquad (2.16)$$

while in the same way we have

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta v}{\delta x} = \lim_{\delta x \to 0} \frac{v(x + \delta x) - v(x)}{\delta x}. \qquad (2.17)$$

Now put $f(x) = u(x) + v(x)$ into the last term of (2.12) to get

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{u(x+\delta x) - u(x)}{\delta x} + \frac{v(x+\delta x) - v(x)}{\delta x},$$

noting that the function $f(...)$ is the sum of the two parts $u(...)$ and $v(...)$, whether the three dots (the argument of the function) stand for $x + \delta x$ or x. Then go to the limit $\delta x \to 0$ in each of the three terms: the first one, from (2.15), gives you $\mathrm{d}y/\mathrm{d}x$; the next one gives $\mathrm{d}u/\mathrm{d}x$ – from (2.16) – and the last one gives $\mathrm{d}v/\mathrm{d}x$ – from (2.16). In other words,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x} \qquad (2.18)$$

– **the derivative of a sum is the sum of the derivatives.**

Now look at a more general function, $f(x) = au(x) + bv(x)$, where a, b are just constants, and go through the same kind of reasoning. You'll find

$$y = au + bv: \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = a\frac{\mathrm{d}u}{\mathrm{d}x} + b\frac{\mathrm{d}v}{\mathrm{d}x}. \qquad (2.19)$$

The operation of differentiating is said to be **linear**: when applied to a *sum* of two or more functions it gives the sum of the results obtained by treating each function separately.

(ii) Differentiating a product; $y = uv$

Again we can use the definitions in (2.15), (2.16), and (2.17); but now, instead of $y = f(x) = u(x) + v(x)$, we have $y = f(x) = u(x) \times v(x)$. So when x increases by δx the function $u(x)$ will change to $u(x+\delta x)$ and $v(x)$ will change to $v(x+\delta x)$. The fraction

$$\frac{f(x+\delta x) - f(x)}{\delta x}$$

will then become

$$\frac{u(x+\delta x)v(x+\delta x) - u(x)v(x)}{\delta x}$$

and we can make this look simpler by remembering that $u(x+\delta x)$ just means the increased value of $u(x)$, after the change $x \to x + \delta x$ – which we have called $u + \delta u$; and similarly $v(x+\delta x)$ just means $v + \delta v$.

The fraction can then be written more briefly as

$$\frac{(u+\delta u)(v+\delta v) - uv}{\delta x} = \frac{(u\delta v + v\delta u + \delta u\delta v)}{\delta x},$$

where we've just multiplied out the product in the numerator and subtracted the term uv. So the fraction whose limit we are going to look for is a sum of three terms:

$$u\frac{\delta v}{\delta x}, \qquad v\frac{\delta u}{\delta x}, \qquad \frac{\delta u\delta v}{\delta x}. \tag{2.20}$$

When we go to the limit as $\delta x \to 0$, the first and second terms together give us

$$u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

– where we've used the definitions in (2.16) and (2.17). But the third term gives

$$\lim_{\delta x \to 0} \frac{\delta u \delta v}{\delta x}$$

and this can be written as either

$$\frac{\mathrm{d}u}{\mathrm{d}x}\delta v, \quad \text{or} \quad \delta u\frac{\mathrm{d}v}{\mathrm{d}x}.$$

In either case, the result contains a factor (δv or δu) – which must tend to zero as $\delta x \to 0$, since the *ratio* $\delta v/\delta x$ or $\delta u/\delta x$ is a finite number. So we can forget about the third term in (2.20) and keep only the first two. In the limit, then, we are left with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{2.21}$$

To summarize: If we know the derivatives of two functions we can write down the derivative of their product; it is the first function times the derivative of the second plus the second function times the derivative of the first. Note also that if one of the functions is just a constant (e.g. $u = c$, so $y = cv$) there will be only one term, since $\mathrm{d}c/\mathrm{d}x = 0$: thus, for $y = cv$, $\mathrm{d}y/\mathrm{d}x = c\mathrm{d}v/\mathrm{d}x$ – the differentiation doesn't touch the constant factor.

(iii) differentiating a 'function of a function'; $y = f(u),\ u = u(x)$

Now suppose we think of $f(u)$ as a function of the variable x – which it *must* be, because u has one value (and one value only) for any chosen value of x and that value therefore determines $y = f(u) = g(x)$. Notice that we've given the function (the 'rule' for getting the value of y) a new name, writing it $g(x)$, because it's not the same rule used in getting $y = f(u)$. Often the name of the thing we're calculating is also used as the function name, putting $y = y(x)$ and depending on the x or the u to tell us which function we're thinking of; sometimes that's good enough, but here we have to be more careful or we'll get mixed up.

If x goes to $x + \delta x$, u will change to $u + \delta u$ and we can define $\mathrm{d}u/\mathrm{d}x$ as the limit of the ratio $\delta u/\delta x$. We can also define $\mathrm{d}y/\mathrm{d}u$ as the limit of a ratio $\delta y/\delta u$; but what we want is $\mathrm{d}y/\mathrm{d}x$ – the rate of change of y with respect to x, not u. How can we get it? We must use a trick, thinking first of the small, but finite, changes before we go to the limit.

As long as the changes are finite, the *approximate* rates of change, $\delta y/\delta u$ and $\delta u/\delta x$ are simply *fractions*; so it is true that

$$\frac{\delta y}{\delta u}\frac{\delta u}{\delta x} = \frac{\delta y}{\delta x},$$

since the factors δu in the numerator and the denominator on the left cancel. But *now* we can go to the limit as δx, δu, δy all become indefinitely small. The two sides of the equality stay equal but are now expressed in terms of derivatives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{2.22}$$

This process, in which the function we really want, $y = g(x)$, is written more easily in terms of another variable u, is usually called "changing the variable" or "substitution" (of one variable in place of another). In later chapters, and in the Exercises, we'll find many examples of how it works.

With the three methods above, you can differentiate any function you may be given – you'll need nothing more!

Exercises

1) The function plotted in Fig.13 is in fact

$$y = 2 + \frac{4x}{3} - \frac{x^2}{9}.$$

Plot the curve for yourself, using any units you wish, with x going from 0 to 6 units (e.g. 1cm). Now look at the same curve in Fig.14 and calculate

(i) the values of y at Point P, where $x = 1$, and Q, where $x = 4$

(ii) show that the point (X, Y), with $X = Y = 6$ also lies on the curve

(iii) calculate the slope of the straight line PQ, connecting P and Q, which is called a 'chord'

2) Now find an *approximate* value of the slope of the curve in Fig.14, *at the point* P, using the same method as on p.18 (for the function $y = x^2$). Compare this value with that for the slope of the chord PQ – is it a *good* approximation? Show that near the point (6,6), which is a 'maximum' (highest point), the slope seems to be zero.

3) Use the method explained in Section 2.3 to find the derivatives of the functions (i) $y = ax$, (ii) $y = bx^2$, (iii) $y = cx^3$ (a, b, c being constants). (*Hint*: use the definition (2.6) of $\mathrm{d}y/\mathrm{d}x$.)

4) Use the results from the last Exercise, along with (2.19), to find the derivative ($\mathrm{d}y/\mathrm{d}x$) of the function in Exercise 1. Then use your expression to obtain *exact* values of the slopes at points P, Q, and the end-point (X, Y).

5) Use the definition (2.6) to find the derivative of the

function $y = x^{-1}$. (*Hint*: use

$$\delta y = \frac{1}{x + \delta x} - \frac{1}{x}$$

and bring the fractions to a common denominator $x(x+\delta x)$. Find $\delta y/\delta x$ and then the limit of this ratio as $\delta x \to 0$.)

6) Now find the derivative $\mathrm{d}y/\mathrm{d}x$ of each of the following functions

- $y = x(1 - x)$ (*Hint*: This is a product of $u = x$ and $v = (1 - x)$)
- $y = (1 - x)^2$ (*Hint*: This is a function of $u = (1 - x)$)
- $y = x(1 - x)^2$ (*Hint*: This is a product of results you already have)
- $y = x/(1 - ax)$ (*Hint*: Again a product of results you already have)
- $y = x/(1 + ax)$ (*Hint*: Again a product of results you already know)

7) Show that if $y = u/v$ (a *quotient* of two functions) then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{v^2}\left(v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}\right).$$

8) Now invent some functions of your own and make sure you can find their derivatives.

9) In Section 2.4 we saw that the *second derivative*, $\mathrm{d}^2y/\mathrm{d}x^2$, of a function $y = f(x)$ could tell us a lot about the form of the function. At a point where $\mathrm{d}y/\mathrm{d}x = 0$ the function has reached a **turning point**: if increasing x leads to a *decrease* in the slope $\mathrm{d}y/\mathrm{d}x$ then y has reached a *maximum* value and is beginning to fall – the turning point is the 'top of the hill' and $\mathrm{d}^2y/\mathrm{d}x^2$ is *negative*; but if $\mathrm{d}y/\mathrm{d}x$ was already negative, before the turning point, and increasing x leads to an *increase* in the slope, then $\mathrm{d}^2y/\mathrm{d}x^2$ is positive and the turning point was the 'bottom of a valley'. In summary,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \text{ negative}, \qquad y = \text{ maximum} ;$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \text{ positive}, \qquad y = \text{ minimum} .$$

You'll find a simple example of a curve showing both a maximum and a minimum if you make a graph of the function $y = x(x-3)^2$. The function and its first and second derivatives are:

$$y = x(x-3)^2 = x^3 - 6x^2 + 9x,$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + 9, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 12.$$

Now check the following:

- $\mathrm{d}y/\mathrm{d}x = 0$ when $x = 1$ or $x = 3$

- y has a maximum value when $x = 1$

- y has a minimum value when $x = 3$

- When $x = 2$, $\mathrm{d}y/\mathrm{d}x \neq 0$, but $\mathrm{d}^2y/\mathrm{d}x^2 = 0$.

In the last case, the point corresponding to $x = 2$ is called a **point of inflexion**: draw the graph of y against x to see what that means.

10) Find the first, second, and third derivatives of the function

$$y = f(x) = 3x^4 - 4x^3 + 1$$

and show that it has stationary points at $x = 0$ and at $x = 1$. What kind of stationary points does the curve have in these two cases (one of them is also a point of inflexion).

11) Sketch the function $y = x^2 e^{-ax^2}$ (a = constant) for positive values of x and find its first and second derivatives. (You can look ahead to the next Chapter, where you'll find in (3.24) that $(\mathrm{d}/\mathrm{d}x)e^x = e^x$. The rest you can do using the results in Section 2.4. Then find the value of x for which the function reaches its maximum value. What will happen to the position and height of the peak if you double the value of the constant a?

12) Find the first and second derivatives of the two functions

$$y_1 = (1 + x^2)^{-2}, \qquad y_2 = \exp(-x)/(2 - x^2)$$

and then those of their product $y = f(x) = y_1 y_2$.
Sketch the function $y = f(x)$ and look for any turning points in the range $x = 0$ to $x = 2$.

Chapter 3

Some standard derivatives and integrals

3.1 Differentiating the function $y = f(x) = x^n$

This is the simplest of all the functions we might want to differentiate or integrate. We've already met two important examples, with $n = 1$ and $n = 2$, in Section 2.2; but now let's think about the more general case where n is *any* positive integer. Later we'll be able to get similar results when n takes values that are non-integral and even negative.

The method to be used is always the same, starting from the definition in Section 2.3 (equation 2.9). Thus $\mathrm{d}y/\mathrm{d}x$ is the limit as $x \to 0$ of the fraction

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}.$$

The first thing we need is an expression for any positive power of $(x+\delta x)$: this may be written $(a+b)^n$, where $a = x, b = \delta x$, and we want to *expand* this function by writing it in the form

$$(a+b)^n = (a+b)(a+b)(a+b)...\,(a+b) \qquad (n \text{ terms}) \tag{3.1}$$

and multiplying it out. For $n = 2$ we know the answer: $(a+b)^2 = a^2 + 2ab + b^2$, but how do we get the answer for any value of the positive integer n?

Clearly there will be just one term a^n, in which we've taken an a from each factor and multiplied them all together; and there'll be a term b^n, arising when we take b instead of a. So $(a+b)^n = a^n + \; ..... \; + b^n$, where the terms in between will all include both as and bs. To find the missing terms we need to think of them one by one, asking how each one can arise. For example, there will be many terms which contain only one b factor, along with $(n-1)$ a-factors, giving results such as $baaa\;...\;a$, $abaa\;...a$, and so on; and since the order of the factors in a product doesn't matter all the n terms will have the same value $a^{n-1}b$ – giving altogether $n \times a^{n-1}b$ as the second term in the expansion of (3.1).

To get the third term we take *two* b-factors, getting $bbaa \, ... \, a$, $baba \, ... \, a$, $baab \, ... \, a$, etc. where there's always a b (from the first factor in (3.1)) in the first place, but the second b comes first from the second factor and next from the third factor, and so on. But having taken the first b from the first factor in (3.1) there will $n-1$ factors left, from which to take another b; so, since the first b could be taken from *any* of the n factors (not only the first) it may seem that the next term in the expansion would be $n(n-1)a^{n-2}b^2$ – with $n-2$ a-factors and 2 b-factors. But wait a minute! When we take out a b and another b from any two factors, it doesn't matter which we call the 'first' and which the 'second' – they give only one term. So the number we've counted has to be divided by 2.

The first three terms in the expansion of (3.1) can now be written down:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \; .... \quad (3.2)$$

and this is all we'll need. This is called a **binomial expansion**, 'binomial' meaning there are two terms in the function $a+b$ whose nth power is being expanded.

To differentiate $y = f(x) = x^n$ is now easy. The increase in $f(x)$ when $x \to x + \delta x$ follows on using (3.2) with $a = x$ and $b = \delta c$:

$$\begin{aligned}\delta y &= f(x + \delta x) - f(x) \\ &= (x + \delta x)^n - x^n \\ &= x^n + nx^{n-1}\delta x + \tfrac{1}{2}n(n-1)x^{n-2}\delta x^2 + \ \ldots \ - x^n.\end{aligned}$$

The ratio $(\delta y/\delta x)$ in (2.9) thus becomes

$$\frac{\delta y}{\delta x} = nx^{n-1} + \tfrac{1}{2}n(n-1)x^{n-2}\delta x + \ \ldots$$

and in the limit $\delta x \to 0$ the last term goes to zero, so

$$y = x^n : \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = nx^{n-1}. \qquad (3.3)$$

This is the result we wanted: it gives the derivative of x^n for any value of n, provided it is a positive integer.

But what if n is *not* a positive integer? For example, if we want to differentiate the function $f(x) = 1/x$ this will be x^n with $n = -1$. In this case we can use the 'building up' rules in Section 3.2 to go from what we know, the derivative of x^n, to the derivative of what we don't know, x^{-n}. The rule for dealing with a product of two functions, u and v, is

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{dv}}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

and if we put $u = x^n$ (the thing whose derivative we know), taking $v = u^{-1}$, then we can say $uv = 1$ – a *constant*, with derivative (slope) zero. In that case the last equation becomes $0 = u \times (\mathrm{d}v/\mathrm{d}x) + v \times (\mathrm{d}u/\mathrm{d}x)$, or $(\mathrm{d}v/\mathrm{d}x) = -u^{-2}(\mathrm{d}u/\mathrm{d}x)$; so we can rearrange things to get (Remember the rules for handling powers, in Book 1 Section 4.2)

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -x^{-2n} \times (nx^{n-1}) = -nx^{-n-1}.$$

In other words, to differentiate a negative power of x, namely x^{-n} (with n a positive integer) we simply multiply by the power (or 'exponent') $-n$ and reduce the power of x by 1. This is exactly the same rule we obtained in (3.3) except that we replace the positive integer n by the negative integer $-n$. In Book 1, we started with counting – the positive integers – and then defined the negative integers and the zero, so that the same rules still applied; and then went on to talk about 'rational fractions' of the form p/q (p, q positive integers); and finally went on to talk about numbers in the 'field' of *all* real numbers. Now we're once again generalizing; and finding that the same rules apply for differentiating negative powers as applied for positive powers. The next step is to show that things work in exactly the same way even when n is not an integer but instead a fraction $n = p/q$ (positive or negative). Try to prove that the rule (3.3) is still true in this case. Then you'll be able to differentiate things like $y = \sqrt{x} = x^{\frac{1}{2}}$. The final step is to prove that the same rule holds good for all real numbers; that's more difficult but we'll be able to prove it later.

3.2 Integrating the function $y = f(x) = x^n$

In the last Chapter we saw that the process of getting a new function by differentiating could be reversed: differentiating $f(x)$ means, in terms of the graph of $y = f(x)$, finding the *slope* of the curve at any point (x, y); integrating $f(x)$ means finding a new function $F(x)$ such that the slope of $F(x)$ will take us back to the given function $f(x)$. If we use the notation $F'(x) = \mathrm{d}F/\mathrm{d}x$ the problem of integration is to find $F(x)$, knowing that $F'(x) = f(x)$, where $f(x)$ is the given function. The result is denoted by $F(x) = \int f(x)\mathrm{d}x$.

For simple functions like $y = x^n$ we can get the integral just by reversing the rule (3.3): instead of multiplying the function by n (the exponent) and then reducing the value of n by 1, we *increase* the value of n by 1 and then *divide* the result by the (new) exponent. Why must we change n first and then use it in doing the division? It's just common sense: you put your socks on before you put your shoes on, so you must take your shoes *off* before you take your socks off! The operation of taking something off is the **inverse** of putting it on; so to get the inverse of doing both operations, one after the other, you must change the order in which you do the inverse operations. (We've met this idea before in Book 1 Section 6.1, in talking about symmetry operations, and it's a very general idea.) Thus (as long as $n \neq -1$)

$$\text{Given } y = x^n = F'(x) : \text{ then } F(x) = (n+1)^{-1}x^{n+1}, \tag{3.4}$$

which you can check by doing the inverse operations, in the reverse order, on $F(x)$; reduce the $n + 1$ in the exponent to n and *multiply* what you get by $n + 1$ – obtaining x^n, which is the given function $f(x)$. But wait a minute! Why did we put ($n \neq -1$) before writing equation (3.4)? Only because, for that very special case, the rule doesn't work: for $n = -1$ the function $F'(x)$ is $f(x) = 1/x$, which was plotted in Fig.3 and gave us a hyperbola. The integral $F(x)$ then comes out as $F(x) = x^0/0$, which is infinite for *any* value of x and so doesn't even define a function of x. We'll come back to this special case in Section 3.4.

For the moment we can summarize what we've done by using A to mean "multiply the function by the number in the exponent" and B to mean "decrease the number in the exponent by unity". If we call the result D, then $\mathsf{D}x^n = \mathsf{BA}x^n$, describes the operation A *followed by* B –

operators always working on whatever function stands on their *right*. Thus

$$\mathsf{D}(x^n) = \mathsf{BA}(x^n) = \mathsf{B}(nx^n) = nx^{n-1}$$

while (with A^{-1} meaning "*divide* by the exponent" and B^{-1} meaning "increase the exponent by 1")

$$\mathsf{D}^{-1} = \mathsf{A}^{-1}\mathsf{B}^{-1}(x^n) = \mathsf{A}^{-1}(x^{n+1}) = (n+1)^{-1}x^{n+1}a.$$

So everything works as it should: you can get the integral either by reversing the rule (if you have one!) for differentiating a function of some given kind – using ordinary language – or you can do the same thing in symbols. The symbols D, for differentiating (applying $\mathrm{d}/\mathrm{d}x$), and D^{-1}, for integrating (doing the integration denoted by $\int$), were already used in equation (2.11) and are often useful. They were first used long ago by the great German mathematician Leibnitz (1646-1716), who was Newton's rival in developing the calculus, and are nowadays much used in mathematical physics.

3.3 The trigonometric or 'circular' functions

Sometimes the independent variable x is an angle and y is given by a function $f(x)$ such as $\sin x$, $\cos x$, or $\tan x$, which involve the lengths of the sides in a triangle. So as not to get mixed up let's use (just for now) θ instead of x as the name of the angle. If r is the length of a line OP from the origin of coordinates to the point P(x, y), making an angle θ with the x-axis, then $\tan\theta = y/x$ is the slope of the line, while $\sin\theta = y/r$ and $\cos\theta = x/r$. Functions of this kind are called **trigonometric** functions (used in describing the angles in a triangle – *gonos* being the Greek word for 'angle'). The three functions are related by

$$\tan\theta = \frac{y}{x} = \frac{y}{r}\left(\frac{x}{r}\right)^{-1} = \frac{\sin\theta}{\cos\theta} \qquad (3.5)$$

so we only need study $\sin\theta$ and $\cos\theta$. Another useful relationship is

$$\sin^2\theta + \cos^2\theta = 1, \qquad (3.6)$$

which you can easily show from the definitions. (Notice that, for example, $(\sin\theta)^2$ is usually written as $\sin^2\theta$, called "sine-squared-theta".)

The sine and cosine functions first came up in Book 1 (Chapter 4), where we found they could be expressed as **series**. Going back to the usual names (x, y) for the independent and dependent variables, the first few terms in each series have been written out in (1.6). Even though the series are both infinite, we know from (2.16) that the derivative of a sum is the sum of the derivatives;

so we can differentiate term-by-term to find, using the result (3.3) for powers of x,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = 1 - \frac{3x^2}{3.2.1} + \frac{5x^4}{5.4.3.2.1} + \ldots = 1 - \frac{x^2}{2.1} + \frac{x^4}{4.3.2.1} + \ldots$$

and in the same way

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = 0 - \frac{2x}{2.1} + \frac{4x^3}{4.3.2.1} + \ldots = -x + \frac{x^3}{3.2.1} + \ldots .$$

The two results together give us

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x. \tag{3.7}$$

The derivative of $\tan x$ can now be obtained using the rules in Section 2.4, with $\tan x = \sin x/\cos x$.

Let us put $y = uv$, with $u = \sin x$, $v = (\cos x)^{-1}$. Then, using (2.18),

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sin x\frac{\mathrm{d}v}{\mathrm{d}x} + (\cos x)^{-1}(\cos x) = \sin x\frac{\mathrm{d}v}{\mathrm{d}x} + 1, \tag{3.8}$$

by the first result in (3.7). All we need now is the derivative of $v = (\cos x)^{-1}$, which is a 'function of a function' and can be handled using (2.19). If we now call $\cos x$ by the new name w, we can say

$$v = w^{-1}, \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}x}.$$

But we know $\mathrm{d}w/\mathrm{d}x = -\sin x$, from the second result in (3.7), and we also know that the rule (3.3) holds even when n is a *negative* integer; so we can get everything we need.

Thus, using (3.3) with $n = -1$ and new names for the variables (w, v in place of x, y), it follows that $\mathrm{d}v/\mathrm{d}w = (-1)w^{-2}$; and from this

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}w} \times \frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\sin x}{(\cos x)^2}.$$

This is the end of the story! Putting it in (3.8) we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{(\sin x)^2}{(\cos x)^2} + 1 = \frac{(\sin x)^2 + (\cos x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2},$$

– another standard result, usually written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \quad (\sec x = 1/\cos x). \qquad (3.9)$$

(The name "secant" ('sec' for short) is a term used in geometry – enough to know that it's the reciprocal of the cosine! The reciprocals of all three functions $\sin x, \cos x, \tan x$ are also known by their other names, $\mathrm{cosec} x, \sec x), \cot x$ but you won't use them much.)

Just to be more complete, we add the result for the 'cotangent' $\cot x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\mathrm{cosec}^2 x \quad (\mathrm{cosec} x = 1/\sin x), \qquad (3.10)$$

which can be found in the same way as for $\tan x$.

From the formulas for the derivatives we can easily obtain the corresponding integral formulas, just by turning them round. Thus, using $F'(x)$ to denote the function obtained by differentiating any $F(x)$, we suppose the result $F'(x) = f(x)$ is known and can then say

$$f(x) = \mathsf{D}F(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x), F(x) = \mathsf{D}^{-1}f(x) = \int f(x)\mathrm{d}x.$$

If we take $F(x) = \cos(x)$, then we know from (3.7) that $\mathsf{D}\cos(x) = \sin(x)$ and therefore $\cos(x) = \mathsf{D}^{-1}\sin(x) = \int \sin(x)\mathrm{d}x$; and similarly for the sine.

To summarize:

$$\text{Given } \cos x = F'(x): \text{ then } F(x) = \sin x = \int \cos x \mathrm{d}x, \tag{3.11}$$

$$\text{Given } \sin x = F'(x): \text{ then } F(x) = -\cos x = \int \sin x \mathrm{d}x, \tag{3.12}$$

$$\text{Given } \sec^2 x = F'(x): \text{ then } F(x) = \tan x = \int \sec^2 x \mathrm{d}x. \tag{3.13}$$

and finally

$$\begin{aligned} \text{Given } \operatorname{cosec}^2 x &= F'(x) : \\ \text{then } F(x) &= -\cot x = \int \operatorname{cosec}^2 x \mathrm{d}x. \quad (3.14) \end{aligned}$$

3.4 The exponential and logarithmic functions

In Book 1, Section 5.1 we met a number defined as the **limit** of a **series** (remember the shorthand used in Book 1: $2! = 1 \times 2,\ 3! = 1 \times 2 \times 3$, etc,, with $0!$ *defined* as 1, and that $n!$ is read as "factorial n"). If we *define* $n! = 0$, the function $y = e^x$ is a sum of terms $x^n/n!$, starting with $n = 0$:

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \ = f(x), \qquad (3.15)$$

when the number of terms becomes infinite. The result depends on the value we give to x and is denoted here by $f(x)$: it is a function of the independent variable x.

In Book 2 we came across this function again, finding some of its amazing properties. Let's recall them: if you multiply two such series together, taking two different values of $x - x = p$ in one series and $x = q$ in the other

– you find

$$\begin{aligned} f(p)f(q) &= \left(1 + p + \frac{p^2}{2!} + \ldots\right)\left(1 + q + \frac{q^2}{2!} + \ldots\right) \\ &= 1 + (p+q) + \left(\frac{p^2}{2!} + pq + \frac{q^2}{2!}\right) + \ldots \\ &= 1 + (p+q) + \frac{(p+q)^2}{2!} + \ldots\,, \end{aligned} \tag{3.16}$$

where we show terms only up to the 'second degree' (i.e. those with not more than *two* variables multiplied together). The result seems to be just the same function (3.15), but with the new variable $x = p + q$. And if you go on, always putting together products of the same degree, you'll find the next terms are

$$(p+q)^3/3! = (p^3 + 3p^2q + 3pq^2 + q^3)/3! \quad \text{(3rd degree)}$$

and

$$(p+q)^4/4! = (p^4+4p^3q+6p^2q^2+4pq^3+q^4)/4! \quad \text{(4th degree.)}$$

As you can guess, if we take more terms we're going to get the result

$$f(p)f(q) = 1+(p+q)+\frac{(p+q)^2}{2!}+\frac{(p+q)^3}{3!}+\ldots\ = f(p+q). \tag{3.17}$$

To prove this result generally is quite hard: you have to look at all possible ways of getting products of the nth degree (n factors at a time) and then show that what you get can be put together in the form $(p+q)^n/n!$. So for the moment we'll just accept (3.17) as a basic property of the function defined in (3.15): it is called the **exponential function** and is often written as "$\exp x$".

From (3.17) we find, by putting $p = q = x$, that $f(x)^2 = f(2x)$; and on doing the same again $f(x)^3 = f(x) \times f(2x) = f(3x)$. In fact

$$f(x)^n = f(nx). \qquad (3.18)$$

This second basic property lets us *define* the nth power of a number even when n is *not an integer*; it depends only on the series (3.15) and holds good when n is any kind of number (irrational or even complex – look back at Book 1 if you've forgotten what this means). Even more amazing, both (3.17) and (3.18) are true whatever the symbols (x, p, q) may stand for – as long as they satisfy the usual laws of combination, including $qp = pq$ (so that products can be re-arranged, as in getting the result (3.17)).

In Book 1, Section 1.7, the (irrational) number obtained from (3.15) with $x = 1$ was denoted by e:

$$e = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\ \dots \quad = 2.718281828\dots \qquad (3.19)$$

and this gives us a 'natural' *base* for defining all real numbers. From (3.18), $e^n = f(n)$ is true for any n – not just for whole numbers but for *any* number. So changing n to x gives

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots , \tag{3.20}$$

which is the function $f(x) = \exp(x)$ we started from; but we can now think of it as a *power* of the special number e. Any number y can be written as e raised to the power x; $y = e^x$ where e is the base and x is the **exponent**. The 'laws of indices', set up in Book 1 Chapter 4 – but only for indices (powers) that were whole numbers – can now be written in general form as

$$e^x e^y = e^{x+y}, \qquad (e^x)^y = e^{xy}. \tag{3.21}$$

(Notice that the other notation, $e^x = \exp(x)$ is convenient when x may stand for some expression too big to be written as an index; and (3.21) can just as well be written as $\exp(x)\exp(y) = \exp(x+y)$ etc.)

Let's get back to the differential calculus! The one thing that gives the exponential function all its remarkable properties is very simple: if we plot a graph of $y = e^x$, a in Fig.4 (Section 1.3), and draw a tangent to the curve at any point (x, y), we find the slope at that point is exactly equal to the value of y. As an equation,

$$y = \exp x : \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = y = \exp x. \tag{3.22}$$

This is called a **differential equation** – generally a relationship between a function and its derivatives – and this is just about the simplest one you can imagine. Other examples of differential equations appear in almost all parts of science – and will therefore be found in many other books of the Series. Even in the present book we noted (Section 1.4) that the exponential function described the growth of a population: the number of people (N) in a city, or country, increasing at a rate proportional to the number already there – which means $\mathrm{d}N/\mathrm{d}t = cN$, where c is a proportionality constant.

To check that the exponential function really does have the property (3.22) it's enough to use the definition (3.15), differentiating term-by-term: thus

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ ...\right) \\ &= \left(0 + 1 + \frac{2x}{2.1} + \frac{3x^2}{3.2.1} + \ ...\right) \\ &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ ...\right) \\ &= \exp x = y.\end{aligned}$$

In some books you'll find the differential equation is taken as the starting point and the series (3.15) is found as its solution; but however you define $\exp x$ it's a very remarkable function.

The integral of $\exp x$ follows in the usual way, by reversing the rule for getting the derivative (as in (3.11), for example). Thus,

$$\text{Given } \exp x = F'(x): \text{ then } F(x) = \exp x = \int \exp x \mathrm{d}x. \tag{3.23}$$

The logarithmic function

This function has been introduced already, in equation (1.8) of Section 3.1, as the inverse of $\exp x$, and the definition is repeated here:

$$\text{Given} \quad y = \exp x \quad \text{then} \quad x = \log y. \tag{3.24}$$

Fig.7 shows how the function looks, when x-values are plotted vertically and y-values horizontally. If we use the usual convention, plotting y upwards (as the dependent variable) and x left-to-right (as independent variable), then the definition of the **logarithmic function** becomes (exchanging the variables in the last equation)

$$\text{Given} \quad x = \exp y \quad \text{then} \quad y = \log x. \tag{3.25}$$

Now there is an important, but simple, relationship between the derivative of a function and the derivative of its inverse: if you think of y as a function of x, then the derivative $\mathrm{d}y/\mathrm{d}x$ is the limiting value of the ratio $\delta y/\delta x$; but, thinking of x as a function of y, the derivative $\mathrm{d}x/\mathrm{d}y$ is the limit of $\delta x/\delta y$. The product of these two fractions is *unity*, however small δx and the related δy may become; so in the limit

$$\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} = 1, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}. \tag{3.26}$$

We can now go back to (3.25), where $x = \exp y$ requires

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \exp y = x,$$

and using (3.26) it follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}. \tag{3.27}$$

To summarize, we have found the function $y = \log x$ whose derivative is x^n with $n = -1$. This is the 'special case' in which the rule for differentiating the function $f(x) = x^n$ can't be used to get the integral $F(x) = \int f(x)\mathrm{d}x$. So the mystery is solved:

$$\text{Given } y = f(x) = x^{-1} : \text{ then } F(x) = \int \frac{1}{x}\mathrm{d}x = \log(x). \tag{3.28}$$

This is the last of the standard results we set out to find. It may seem strange that we haven't come out with a nice series for $\log x$, something like the one we started from for $\exp x$. Instead, the function is defined as an integral, which isn't easy to evaluate by simple arithmetic. The real reason is that the function we're integrating, $y = x^{-1}$, describes a hyperbola of the form shown in Fig.3; it breaks into two branches, separated by a *singularity* at $x = 0$ where the function and its derivatives become infinite. We can't find a series of the type $y = a+bx+cx^2+dx^3+ ...$, whatever values we give the coefficients $a, b, c, ...$ because at $x = 0$ everything 'blows up'. This just shows how careful you must be in mathematics if you don't want nasty surprises! It's always a good idea to plot the functions you're dealing with, to 'see' how they behave.

A note on changing the variable

We now have a short list of standard functions, whose derivatives and corresponding integrals can be taken as 'known'. But the list can be greatly extended by using the rules in Section 2.4. Many examples will be found in the Exercises. Here we just give one to show it's not difficult.

Suppose we want the derivative of $y = \exp(ax)$ where $a \neq 1$. We can think of ax as a new variable, putting $ax = t$ and $y = \exp(t)$. The rule for differentiating a function of a function then gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = a\frac{\mathrm{d}y}{\mathrm{d}t} = a\exp(t) = a\exp(ax)$$

– so easy that, with a bit of practice, you can do it in your head! Do this for all the functions you've looked at in this Chapter and then check your results against those listed in the Table at the beginning of Chapter 4.

Exercises

1) Find a formula for the derivative of the function $y = x^{p/q}$, where p/q is a rational fraction (the ratio of two positive integers). (*Hint*: y is a function of a function, $y = u^p$, where $u = x^{1/q}$)

2) Prove the results in (3.7) and (3.9), starting from the formulas given in Book 1 (end of Chapter 4), namely

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b, \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b. \end{aligned}$$

3) From the results in (3.7) and (3.9), get expressions for the derivatives of the functions $y = \sin ax$, $y = \cos ax$, and $y = \tan ax$.

4) What forms do the results (3.11), (3.12), and (3.13) take when x is replaced by ax?

5) What form does the differential equation given in (3.22) take when x is replaced by ax? And what is the corresponding form of (3.23)?

6) Show that $\log X$ can also be expressed as a *definite* integral (measured by the area under a curve between end points at 1 and X, namely

$$\log X = \int_1^X x^{-1} \mathrm{d}x.$$

(*Hint*: Look back at equation (2.9), where the function being integrated was in that case $f(x) = x$. Here, instead, $f(x) = 1/x$ but the meaning is similar: $f(x)\mathrm{d}x$ is the increase in the 'area function' (now $\log X$) when the upper boundary is increased to $X + \mathrm{d}x$.) Note that choosing the lower boundary at $x = 1$ simply guarantees that $\log 1 = 0$ ($e^0 = 1$).

7) Find the derivatives $\mathrm{d}y/\mathrm{d}x$ of the following functions:

$$
\begin{aligned}
(a) \quad y &= (x + 1/x)^2\,, \\
(b) \quad y &= (1 + x^2)/(1 - x), \\
(c) \quad y &= (1 - \cos x)/(1 + \cos x), \\
(d) \quad y &= x \sin x, \\
(e) \quad y &= x^2 \cos x, \\
(f) \quad y &= \sqrt{1 + x}, \\
(g) \quad y &= \sqrt{1 + \sin x}, \\
(h) \quad y &= 1/\sqrt{1 - x}, \\
(i) \quad y &= 1/\sqrt{1 - x^2},
\end{aligned}
$$

$$(j) \quad y = \sin x/x,$$

$$(k) \quad y = x/\sin x.$$

$$(l) \quad y = e^x \sin x/x,$$

$$(m) \quad y = \exp(-ax^2)\sin x.$$

8) Make rough sketches of some of the functions used in Exercise 7, to see how they behave as $x \to \pm\infty$. Look out for singularities and any other points of interest.

Chapter 4

Integrals – and ways of getting them

4.1 The problem of integration

If someone gives you a function, $y = f(x)$, without telling you anything about it, how can you find the indefinite integral $F(x) = \int f(x)\mathrm{d}x$? All you know is that the *derivative* of $F(x)$ is the given function $f(x)$. That is the problem we face in this Chapter.

In other words, you have to solve the equation

$$\frac{\mathrm{d}F}{\mathrm{d}x} = f(x) \qquad (4.1)$$

– which is a **differential equation** (relating, in general, functions and their derivatives).

The integrals we were able to write down in Chapter 3 were all obtained by 'reversing' the rule for getting the derivative $f'(x)$ of some known function $f(x)$: this rule describes a *direct* operation on the function $f(x)$, denoted at the end of Section 3.2 by the operator D. With this notation,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \mathsf{D}f(x). \tag{4.2}$$

But to find $F(x)$ from (4.1) requires the *inverse* operation, indicated by the **inverse operator** D^{-1}, such that

$$F = \mathsf{D}^{-1}f(x). \tag{4.3}$$

We have recipes, discovered in the last Chapter, for doing the direct operation D on various *known* functions; but we have no recipe for the inverse operation D^{-1}. All we can say is that if D works on the function $F(x)$ it should give, according to (4.1), $f(x) = \mathsf{D}F(x)$. The trouble is that $F(x)$ is not a *known* function – it is the answer we are looking for! We could make a guess at the answer, work on it with D (i.e. differentiate it), and see if the result is the given function $f(x)$. If it *is*, then we guessed right; but there are millions of other guesses we could have made – how do we find the right one (if there *is* one)?

Let's start by making a Table, listing the results we've found so far:

Function $f(x)$	Deriv. $= \mathsf{D}f(x)$	Integral $= \mathsf{D}^{-1}f(x)$
$x^n \ (n \neq -1)$	nx^{n-1}	$\dfrac{x^{n+1}}{n+1}$
$\dfrac{1}{x} \ (= x^{-1})$	$-x^{-2}$	$\log x$
$\log x$	$\dfrac{1}{x}$	$x \log x - x$
e^x	e^x	e^x
$\sin x$	$\cos x$	$-\cos x$
$\cos x$	$-\sin x$	$\sin x$

Table 1: Some derivatives and integrals

Notes on Table 1

The operator D and its inverse D^{-1} should have the property

$$\mathsf{D}\mathsf{D}^{-1} = \mathsf{D}^{-1}\mathsf{D} = \mathsf{I}, \tag{4.4}$$

where I is called the **identity operator** which leaves any function unchanged. Equation (4.4) means that when the two operators, D and D^{-1}, work on a function one after the other, either way round, they should not change the function in any way: each one undoes the work of the other. You should check this property carefully, using the results listed in Table 1 (Don't worry about the line starting with '$\log x$' – we don't yet have the result in Column 3, but you'll find it in Section 3.4) Remember that operators always work on functions that stand on their right, so that $\mathsf{AB}f(x)$ means "operate with B first and then apply A to the result". This is just a 'convention' agreed on in Book 1 (Chapter 7); but sometimes you might find a book that uses the opposite convention, so watch out!

To start you off, take the first line in the Table where the operators work on $f(x) = x^n, \quad (n \neq -1)$. To evaluate $\mathsf{D}\mathsf{D}^{-1}f(x)$ we have

$$\begin{aligned} \mathsf{D}\mathsf{D}^{-1}f(x) &= \mathsf{D}\left(\frac{x^{n+1}}{n+1}\right) \\ &= \frac{1}{n+1}(n+1)x^{n+1-1} = x^n = f(x), \end{aligned}$$

where you'll notice that after the first operation the n has been changed to $(n+1)$ – according to the result in the third column. But if you do the operations in the reverse order you get

$$\begin{aligned} \mathsf{D}^{-1}\mathsf{D}f(x) &= \mathsf{D}^{-1}\left(nx^{n-1}\right) \\ &= n\left(\frac{x^{n-1+1}}{n-1+1}\right) = x^n = f(x), \end{aligned}$$

where after the first operation you had to use $n-1$ (from the second column) in place of n. In both cases numerical multipliers are not touched by the operators and can be moved to the left.

One last, but very important, point. You go from a result in the first column of the Table to the one in the third column, by integrating, and you can change the result (which is called the '*indefinite* integral') by adding to it any *constant* term C – without spoiling anything. This is because the indefinite integral is *defined* only as a function satisfying the equation (4.1); and if you change $F(x)$ into $F(x)+C$ it doesn't make any difference, because $dC/dx = 0$ for any constant. So, even if it's not always written in Tables of integrals, it should be understood. Notice also that if we want only a Table of integrals then we only need the entry in Column 1, which is the *integrand* – the thing we want to integrate – and the corresponding entry in Column 3 – which is the integral. (By now we know how to *differentiate* anything, so in later Tables we'll leave out the derivatives.)

Inverse functions

In Section 3.3 we found the derivatives of the circular functions $\sin x, \cos x$, and $\tan x, \cot x$. We can add more lines to Table 1 by including, along with these functions, their *inverses*: these are called $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$, and $\cot^{-1} x$. The idea of the 'inverse' of a function was introduced in Chapter 1 (Section 1.4); the only new thing we're doing now is to write the inverse of any function $f(x)$ as $f^{-1}(x)$, instead of giving it a new name like, say, g(x). In words, the relationship $y = \sin x$ is simply turned round so that $x = \sin^{-1} y$ means "x is the angle whose sine is y" and similarly for the others.

To make sure you don't get mixed up with something to the *power* -1, other names are sometimes used: for example "arcsin" x instead of $\sin^{-1} x$, but we'll go on with the "-1" notation, which reminds you that it's an inverse function. Remember also that when x is used as a variable it is a *number of radians*: if you turn a pointer so that it points in exactly the opposite direction, then you've turned it through π radians, where $\pi \approx 3.14159$. (If you've forgotten all about angles, look back at Book 2.)

It's quite easy to differentiate the inverse functions by using the simple rule in (3.26): if we have a function

$y = f(x)$ and want to think of x as a function of y, writing $x = f^{-1}(y)$, then

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\mathrm{d}y/\mathrm{d}x} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}. \qquad (4.5)$$

Thus, with $y = \sin x$ and $(\mathrm{d}y/\mathrm{d}x) = \cos x$, we have $x = \sin^{-1} y$ and $(\mathrm{d}x/\mathrm{d}y) = 1/(\mathrm{d}y/\mathrm{d}x) = 1/\cos x$.

But is this really the result we want? We've always used x for the independent variable (plotted along the horizontal axis) and y for the dependent variable (plotted vertically); so to keep this practice we must swap x and y. For the inverse function $y = \sin^{-1} x$ we then write

$$y = \sin^{-1} x : \qquad \mathrm{d}y/\mathrm{d}x = 1/\cos y$$

– but we're still not finished, because the result for $\mathrm{d}y/\mathrm{d}x$ should be written in terms of the independent variable x, not in terms of $\cos y$. However, $x = \sin y$ and we know that for *any* angle, θ say, $\sin^2\theta + \cos^2\theta = 1$; so we can say $\cos^2 y = 1 - \sin^2 y = 1 - x^2$. And putting it all together this gives

$$y = \sin^{-1} x : \qquad \mathrm{d}y/\mathrm{d}x = \frac{1}{\sqrt{1-x^2}}. \qquad (4.6)$$

In the same way (do it!) you find three more results:

$$y = \cos^{-1} x : \qquad \mathrm{d}y/\mathrm{d}x = \frac{-1}{\sqrt{1-x^2}}, \qquad (4.7)$$

$$y = \tan^{-1} x : \qquad \mathrm{d}y/\mathrm{d}x = \frac{1}{1+x^2}, \tag{4.8}$$

$$y = \cot^{-1} x : \qquad \mathrm{d}y/\mathrm{d}x = -\frac{1}{1+x^2}. \tag{4.9}$$

From these derivatives, $\mathsf{D}F(x) = f(x)$, we can write down the corresponding indefinite integrals, $\mathsf{D}^{-1}f(x) = F(x)$, and extend the list in Table 1. Thus, from (4.6), the new integral formula will be

$$\mathsf{D}^{-1}(\mathrm{d}y/\mathrm{d}x) = \mathsf{D}^{-1}\frac{1}{\sqrt{1-x^2}} = y = \sin^{-1} x.$$

Similarly, the other formulas lead to the Table below:

Function $f(x)$	Integral $= \mathsf{D}^{-1}f(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{\sqrt{1-x^2}}$	$-\cos^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\frac{1}{1+x^2}$	$-\cot^{-1} x$

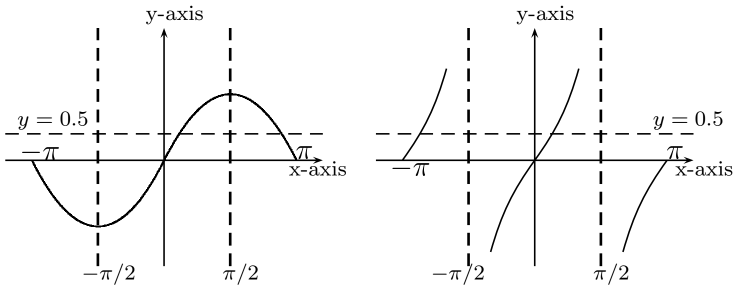
Table 2: Some more indefinite integrals

Notes on Table 2 (Only for the very brave!)

Something looks odd about this Table: it seems you can get two different answers when you integrate the same function! To clear up this mystery you have to look at the forms of the functions you're dealing with. Most of the functions we've looked at so far are *single-valued*: if you're given a value of the independent variable x, then there's only *one* corresponding value of $y = f(x)$. But if x is an angle, then $y = \sin x$, for example, has the same value for angles $x, \pi - x, 2\pi + x$, ..., and so on. This means the inverse function $x = \sin^{-1} y$ (the angle whose sine is y is a *multiple-valued* function of the variable y, as can be seen from Fig.5 in Chapter 1 (repeated here in Fig.16a).

The trouble is that $y = \sin x$ is a *periodic* function, which wiggles up and down forever, in both directions, as x goes to $\pm\infty$. If you look only at the range with x between $-\pi/2$ and $+\pi/2$ (half a complete up-down wiggle – which goes from $-\pi$ to $+\pi$, as shown in Fig.16a), then $y = \sin x$ and its inverse $x = \sin^{-1} y$ are both single-valued functions: $x = \pi/12$, example, means $y = \frac{1}{2}$ for that value and that value only – and *vice versa* when you think of x as a function of y. If you go outside the range indicated, then you can find an infinite number of angles giving $y = \frac{1}{2}$.

To make sure we get every possible pair of related x, y values only once, we consider only the range of x values going from $-\pi/2$ to $+\pi/2$: values of y within this range are called the **principal values** of the function.



(a) $y = \sin x$ (b) $y = \tan x$

Figure 16

The function $y = \cos x$ is also periodic and looks exactly like $y = \sin x$ if you slide it back (to the left) through $\frac{1}{2}\pi$, so that its value starts off at 1 instead of 0. The range in which $y = \cos x$ takes its principal values thus goes from $x = 0$ to $x = \pi$.

On the other hand, the functions $y = \tan x$ and $y = \cot x$ go smoothly from $-\infty$ to $+\infty$, but break into separate 'branches' as shown in Fig.16b. The first branch, for $y = \tan x$, which holds the principal values, falls within the range $x = -\pi/2$ to $x = +\pi/2$; but if you shift the curve left or right, changing x to $x \pm \pi, x \pm 2\pi$, etc. you get an infinite number of parallel branches.

The first branch of the curve $y = \cot x$, looks exactly the same as that for $y = \tan x$, but shifted to the right by $\frac{1}{2}\pi$. And, again. all possible values of $y = \cot x$ are contained, once and once only, in the range $x = -\pi/2$ to $x = +\pi/2$.

Whenever we want to integrate a function we'll start by asking if it can be related to any of the 'standard forms' listed in Tables 1 and 2; and we'll find later that even if sometimes we get more than one result there's nothing to worry about – because we mustn't forget they are *allowed* to look different when the arbitrary constant C is included! To make them agree we only have to choose the right value of C.

So far, in earlier Chapters, we've been thinking of integration just as the inverse of differentiation: D is the operator that takes you from a function $f(x)$ to its derivative $\mathrm{d}f/\mathrm{d}x$ (which is the slope of the curve $y = f(x)$ at any chosen point (x, y); while the inverse operator D^{-1} is the one that takes you back, from the derivative to the function itself, and describes *integration* – putting the function together again. This is called 'indefinite' integration because any constant C can be added on to the result, to give a new function $y = f(x) + C$ with exactly the same derivative, for given x, as $y = f(x)$: the new curve is just like the original but is shifted upwards by an amount C at all points. If we don't bother to put in the constant, then we call $y = f(x)$ the **primitive function**: it gives the correct derivative $\mathrm{d}f/\mathrm{d}x$ (which we were given) but so do all the other functions $f(x) + C$. Which one we choose to call 'primitive' doesn't matter: they are all equally satisfactory as integrals

4.2 But why do we want to integrate anyway?

In Chapter 2, when the idea of integration was first used, we were trying to find the *area* under the curve $y = f(x)$ between the curve, the x-axis, and vertical lines ('ordinates') at $x = x_0$ and $x = X$, as in Fig.15, and the value of the area A depended on where we put the vertical boundaries which mark the integration *limits*. If we choose the limits to be $x = x_1$ and $x = x_2$, the 'upper' and 'lower' limits, this area is a 'definite' integral denoted by $A = \int_{x_1}^{x_2} f(x)\mathrm{d}x$. In particular

$$A = \int_{x_0}^{x} f(x)\mathrm{d}x$$

is the area corresponding to lower limit at $x = x_0$ and upper limit taken at 'any old value' of x.

We also noted, at the end of Section 2.2, that if the upper limit is increased from x to $x + \mathrm{d}x$, by the infinitesimal amount $\mathrm{d}x$, then the area function increases according to

$$\frac{\mathrm{d}A}{\mathrm{d}x} = f(x). \qquad (4.10)$$

But now look back at our definition of indefinite integration, through (4.2) and (4.3), using the area function $A(x)$ in place of $f(x)$, and you see that

$$\frac{\mathrm{d}A}{\mathrm{d}x} = f(x) \quad \text{means} \quad A = \mathsf{D}^{-1}f(x) = \int f(x)\mathrm{d}x \qquad (4.11)$$

In other words, to get the definite integral as an area between two limits, x_1 (lower) and x_2 (upper), you first get the function $A(x)$, by reversing the rules for differentiating, and then find the area as

$$\int_{x_1}^{x_2} dA = A(x_2) - A(x_1) = [A(x)]_{x_1}^{x_2}, \qquad (4.12)$$

where the square-bracket quantity just stands for the difference of the area function between the upper and lower limits.

It's now clear why the arbitrary constant C in an indefinite integral doesn't matter when we need to calculate a definite integral: when you do the subtraction in (4.12) the constants will cancel. Always remember, however, that the function you're integrating must 'behave' well in the range (x_1, x_2); the function $f(x)$ must show no 'breaks' (discontinuities) or 'infinite peaks' (singularities), where the area or its derivative could not be defined.

Some Applications

Some of the most direct applications of the calculus have to do with the calculation of **length, area** and **volume**, and with their rates of change with time. For example, the distance you travel along some curve connecting two points is a path length, s say, measured from the starting point where $s = 0$. And your speed v will be the rate of increase of s with time: $v = ds/dt$, at whatever point you happen to have reached. If the curved path is described, using rectangular coordinates (x, y), by the relationship $y = f(x)$ then both s and v will be functions of the one independent variable x;

and as you go along s, v and x will all be functions of the time t. So we can write (not thinking yet about the time t)

$$y = f(x), \qquad x = g(t), \qquad s = s(x), \qquad v = v(x), \quad (4.13)$$

where we've used f and g as the first two function names, but kept s and v to serve for both the physical quantities and the names of the functions which describe them.

That may seem confusing, but it's the usual convention in Science: we can't invent a new function name every time we want to change the variable; and anyway writing $s = s(x)$ tells us that here we're thinking of s as a function of x, while $s = s(t)$ says that s is also a function of t. The *value* of the dependent variable, on the left-hand side of such equations, is determined by that of the independent variable on the right; and that one-to-one correspondence is what defines the functional relationship. Double use of the same symbol is not strictly correct, but we always know what is meant – and that is what matters!

Now let's travel along the curve shown in Fig.17, starting from P_1, where $x = x_1$, and finishing at P_2, where $x = x_2$.

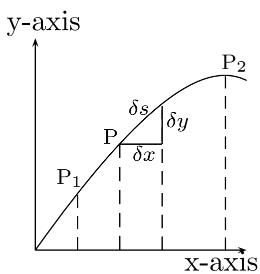


Figure 17

At point $P(x, y)$ on the curve, anywhere between these limits, the next small step will carry you to $(x + \delta x, y + \delta y)$ and the step length will be $\delta s \approx \sqrt{\delta x^2 + \delta y^2}$, provided the increases $\delta x, \delta y$ are small, as in the Figure. (Remember that δx is a single quantity and that δx^2 means its square, $(\delta x)^2$, not the increase of x^2.)

Now the total path length from P_1 to P_2 will be the sum of all the elements δs and we write this as: $s_{12} \approx \sum \delta s$, where

$$\delta s \approx \sqrt{\delta x^2 + \delta y^2} = \sqrt{\delta x^2 \left(1 + \frac{\delta y^2}{\delta x^2}\right)}.$$

In the limit where $\delta x, \delta y \rightarrow 0$ the ratio on the right becomes $(\mathrm{d}y/\mathrm{d}x)^2$ and the sum over all infinitesimal elements becomes an integral:

$$s_{12} \rightarrow \int_{x_1}^{x_2} \left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{1}{2}} \mathrm{d}x. \qquad (4.14)$$

It is a definite integral between the limits $x = x_1$ and $x = x_2$ and this result is general for *any* plane curve.

To see how things work out, look at the quadrant of a circle in Fig.18.

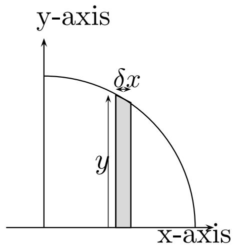


Figure 18

Here the equation of the curve is $x^2 + y^2 = a^2$ (a being the radius). Thus, taking a positive square root (why?) and doing the differentiation,

$$y = \sqrt{a^2 - x^2}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{a^2 - x^2}}.$$

So the whole arc length from P_1 to P_2 will be, from (4.14),

$$s_{12} = \int_{x_1}^{x_2} \left[1 + \frac{x^2}{a^2 - x^2}\right]^{\frac{1}{2}} \mathrm{d}x = \int_0^a \frac{a}{\sqrt{a^2 - x^2}} \mathrm{d}x. \quad (4.15)$$

This is not exactly one of the standard integrals in Table 2, but we can use the same trick as at the end of Section 3.3 to make it so. Let's put $x = au$, where u is a new variable ($u = x/a$ or x in units of the radius) and remember that $\int f(x)\mathrm{d}x = \int f(x)(\mathrm{d}x/\mathrm{d}u)\mathrm{d}u$. Since $\mathrm{d}x/\mathrm{d}u = a$ and $f(x) = f(au)$ in terms of u, the last result can be re-written as

$$s_{12} = \int_0^a \frac{a}{\sqrt{a^2 - a^2u^2}} a\mathrm{d}u = a\int_0^1 \frac{1}{\sqrt{1 - u^2}} \mathrm{d}u. \quad (4.16)$$

This *is* a standard integral; it is $\sin^{-1} u$ taken between the limits $u = 0$ and $u = 1$, which correspond to $x = 0$ and $x = a$ i.e. the ends of the quadrant in Fig.18. On putting in these limits (4.16) gives

$$s_{12} = a(\sin^{-1} 1 - \sin^{-1} 0) = a(\pi/2 - 0) = \tfrac{1}{2}a\pi. \qquad (4.17)$$

The circumference of a whole circle, of radius a, is 4 times the arc length for one quadrant and is thus $2\pi a$ – as we knew from the geometrical argument in Book 2.

As a second application let's calculate the area of the circle. We take the quadrant shown in Fig.18, divided into vertical strips – the one at distance x from the centre having height y, width δx, and area $y\delta x$. If we add together the areas of all the strips, going from $x = 0$ to $x = a$ (the radius of the circle), we'll get the area of the whole quadrant – which will be approximate, if the strips have finite width, or exact if we go to the limit $\delta x \to 0$, with an infinite number of strips. The differential element of area will be $\mathrm{d}A = y\mathrm{d}x = \sqrt{a^2 - x^2}\mathrm{d}x$ and the whole area will then be the definite integral

$$A = \int_0^a \sqrt{a^2 - x^2}\mathrm{d}x. \qquad (4.18)$$

Again, this is not one of the standard integrals in Tables 1 and 2; and the trick we used in (4.15) doesn't work this time (try it!). We'll come back to it later, but for now let's try another way of dividing the area into infinitesimal pieces: we can take them to be *circular* strips as in Fig.19. Each strip, of width δr, will have an area $\delta A=$ (circumference) $\times$

(width); but we just found the circumference to be 2π times the radius, so $\mathrm{d}A = 2\pi r \mathrm{d}r$ for a strip of width $\mathrm{d}r$ and the total area becomes the definite integral

$$A = \int_0^a 2\pi r \mathrm{d}r = 2\pi \left[\tfrac{1}{2}r^2\right]_0^a = \pi a^2, \qquad (4.19)$$

where we've used the first result in Table 1.

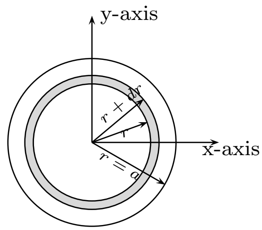
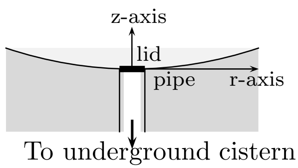


Figure 19

So, if you can't see a way of integrating to get an area, try looking for another way of choosing the differential element of surface area. Rectangular Cartesian coordinates (x, y) don't give the simplest way of dealing with circles, but a radius and an angle (r, θ) clearly do, the equation of the circle being simply $r = a$ instead of $x^2 + y^2 = a^2$.

As a last application we'll calculate a volume. You might want to find how much water a big circular pond can hold and to do that you'll need to know how deep it is. Again it's easiest to use r, distance from the centre, as the independent variable; and if we call the depth of the pond z we can suppose $z = f(r)$. To find the function you can go out in a boat with a long stick, dipping it in the water to find z for a few values of r. If the pond is shallow it may be that one or two values will give a fairly good approximation: in the middle ($r = 0$) the depth might be d, while at the edge ($r = a$, say) it will be zero. A shallow dish like that is often well described by a **parabola**, as in Fig.20 which shows the shape of the bottom.



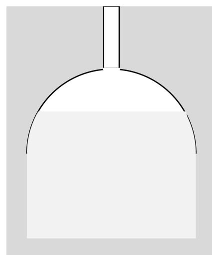


Figure 20 Figure 21

The general equation of a parabola has the form $z = A + Br + Cr^2$, where A, B, C are constants, which must be chosen to fit with what we know. If we put the z-axis pointing upwards, as usual, we know $z = 0$ when $r = 0$; so A must be zero. And if you go out from the centre the depth must be the same at points r and $-r$; so the linear term can't be there – you must put $B = 0$. Finally, at the edge, $z = Cr^2$ must give d when $r = a$; so $d = Ca^2$ and $C = d/a^2$. The equation of the pond bottom is thus

$$z = f(r) = (d/a^2)r^2 \qquad (4.20)$$

and now we can get the volume of water it can hold.

How shall we choose the small elements of volume? If we think of the water as a set of circular slabs, of thickness δz and diameter r, each one will have a volume δz times its surface area; and we've just found that the slab at height z will have an area πr^2. So the differential element of volume will be $\mathrm{d}V = \pi r^2 \mathrm{d}z$, where z is related to the diameter of the element according to (4.20). The differential $\mathrm{d}z$, as we know from p.22, is $\mathrm{d}z = (\mathrm{d}z/\mathrm{d}r)\mathrm{d}r$; and by differentiating (4.20) we find $(\mathrm{d}z/\mathrm{d}r) = (d/a^2)(2r)$. The volume element thus depends on the diameter of the slab:

$$\mathrm{d}V = \pi r^2 (d/a^2) 2r \mathrm{d}r = 2\pi (d/a^2) r^3 \mathrm{d}r. \tag{4.21}$$

When the pond is full to the brim, the volume of water it contains will be the definite integral

$$\begin{aligned} V &= 2\pi(d/a^2)\int_0^a r^3 \mathrm{d}r = 2\pi(d/a^2)\left[r^4/4\right]_0^a \\ &= \tfrac{1}{2}\pi(d/a^2)a^4 = \tfrac{1}{2}\pi(da^2) \end{aligned} \tag{4.22}$$

– a beautifully simple result.

In hot counties where, there isn't much rain, water is very precious and if you've been lucky enough to get a pond full the next problem is how to keep it – because if you just leave it in the pond it will dry up in almost no time. The best way of keeping water fresh is to make a big underground cistern, cutting it out of the rock if you can and lining it with clay so it won't leak; then you can empty the water into it every time the pond gets full. People have been doing this for thousands of years: at Istanbul in Turkey there are some enormous cisterns that were made two thousand years ago by the Romans – and they still hold water!

Fig.21 shows how a cistern might look: the water from the pond in Fig.20 goes down into it through a pipe, when you take the lid off, and to know how big to make it and how many pondfuls it will hold you have to calculate volumes. You can do that for the cistern provided you can make measurements to find the relationship between the diameter r and the height z above the bottom: once you have $r = f(z)$ you can express the volume V up to any water level, Z say, as a definite integral, $\int_0^Z \mathrm{d}V$ just as we did for the shallow pond. Think about it!

4.3 Integration 'by substitution'

In this section we start looking for ways of getting an integral that's not in our list of 'standard' integrals (Tables 1 and 2) – by relating it to an integral that *is*. The first method is called **integration by substitution** and we've already met a simple example of it. At the end of Chapter 3 there was a note on "changing the variables". This gave a useful way of integrating something like, say, $f(ax + b)$, when we only knew the result of integrating $f(x)$. We simply think of $ax + b$ as a new variable, calling it u say, and use the result we know to get $\int f(u)\mathrm{d}u$ – which we can then write in terms of the original x.

The method is based on equation (2.22) in Chapter 2, which tells us how to differentiate a "function of a function", namely $y = f(u), \quad$ where $u = u(x)$. The *inverse* operation, of integrating a function of a function, follows in the usual way: for since y then becomes also a function of x

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

and, on integrating both sides of this equation with respect to x, we get

$$y = \int \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{4.23}$$

In this Section we'll look at examples of how this rule works.

- (i) $y = f(x+a) = (x+a)^n$, $a =$ constant. Denote $x + a$ by u and then use (4.23) to get

$$\begin{aligned}\int (x+a)^n \mathrm{d}x &= \int u^n \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u = \int u^n \mathrm{d}u \\ &= \frac{u^{n+1}}{n+1} = \frac{(x+a)^{n+1}}{n+1},\end{aligned}$$

  since, from (4.5), $(\mathrm{d}x/\mathrm{d}u) = 1/(\mathrm{d}u/\mathrm{d}x) = 1$. So in this case we simply use the result in the Table of integrals, substituting $(x+a)$ in place of the original x. Note that this works for all the functions listed, not just for x^n, since $\mathrm{d}x/\mathrm{d}u$ depends only on the form of $u(x)$.

- (ii) $y = f(ax) = (ax)^n$, $a =$ constant. Now put $ax =$

u and again use (4.23):

$$\begin{aligned}\int f(ax)\mathrm{d}x &= \int f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u = \int f(u)(1/a)\mathrm{d}u \\ &= (1/a)\int f(u)\mathrm{d}u = (1/a),\end{aligned}$$

since $(\mathrm{d}x/\mathrm{d}u) = (1/a)(\mathrm{d}u/\mathrm{d}u) = 1/a$. So we use the result in the Tables, for $\int f(x)\mathrm{d}x$, with ax in place of x, but then divide by the constant a.

- (iii) $y = f(ax+b), a, b$ both constants. Now put $ax + b = u$ and again use (4.23):

$$\begin{aligned}\int f(ax+b)\mathrm{d}x &= \int f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u = \int f(u)(1/a)\mathrm{d}u \\ &= (1/a)\int f(u)\mathrm{d}u.\end{aligned}$$

  So again we can use a tabulated result, with u in place of the x, just dividing it by the constant a.

- (iv) $y = \int f(u)\mathrm{d}u,\ u = u(x) = x^2$. In this case (4.23) tells us that

$$\int f(u)\mathrm{d}u = \int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f(x^2)(2x)\mathrm{d}x;$$

  so the integral on the right can be replaced by that on the left, which may be easier to evaluate. This result is general: whenever the integrand is a function of the new variable u, multiplied by the derivative

$\mathrm{d}u/\mathrm{d}x$, we can substitute the result on the left. Thus, in this example, $\int f(x^2)(2x)\mathrm{d}x = \int f(u)\mathrm{d}u$, but more generally

$$\begin{aligned} \text{Any integral of the form } I &= \int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\ \text{can be replaced by } I &= \int f(u)\mathrm{d}u. \quad (4.24) \end{aligned}$$

An important special case follows on using $f(u) = 1/u$, for then

$$\int \frac{(\mathrm{d}u/\mathrm{d}x)}{u}\mathrm{d}x = \int \frac{1}{u}\mathrm{d}u = \log u. \qquad (4.25)$$

In words, *Whenever the numerator in an integrand is the derivative of the denominator*, the integral is the logarithm of the denominator.

. Of course you don't need to remember all possible cases, as they're so easy to get from (4.23). For example,

- (iv) $y = \sin ax$, $a =$ constant. Again put $ax = u$ and use (4.23):

$$\begin{aligned} \int \sin(ax)\mathrm{d}x &= \int \sin u \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u = (1/a)\int \sin u \mathrm{d}u \\ &= (1/a)(-\cos u) = -(1/a)\cos(ax), \end{aligned}$$

and finally

- (v) $y = \frac{1}{ax+b}$, a, b both constants. Substituting $ax + b = u$,

$$\int \frac{1}{ax+b} \mathrm{d}x = (1/a)\log(ax+b).$$

It's not always easy to choose a substitution that simplifies things: you just have to try anything you can think of that looks as if it might work. For example, when we were trying to find the area of a quadrant, using Cartesian coordinates, we found the result

$$A = \int_0^a \sqrt{a^2 - x^2}\mathrm{d}x$$

but didn't know how to evaluate the integral. If you try $u^2 = a^2 - x^2$, to get rid of the square root, you'll find it doesn't help. But trying $x = a \sin u$ does: it gives, before starting the integration,

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 u} = a\sqrt{1 - \sin^2 u} = a \cos u.$$

With this substitution we'll also need the factor $\mathrm{d}x/\mathrm{d}u$ to put in the integrand. This will be $\mathrm{d}x/\mathrm{d}u = a \cos u$ and now we can do the integration:

$$\int \sqrt{a^2 - x^2}\mathrm{d}x = \int a \cos u \times a \cos u \mathrm{d}u = a^2 \int \cos^2 u \mathrm{d}u,$$

which looks easier. But how do we do it? We need to remember something from Book 2, where we found how to handle trigonometric functions like $\sin x$ and $\cos x$. We found how to get the sine and cosine of a *sum* of two angles, given in equations (4.22) of Chapter 5: these took the forms $\sin A+B = \sin A\cos B + \cos A\sin B$ and $\cos A+B = \cos A\cos B - \sin A\sin B$. And now, by putting $A = B = u$, we can do the integration! It becomes

$$\begin{aligned}\int\sqrt{a^2-x^2}\mathrm{d}x &= a^2\int\cos^2 u\mathrm{d}u = a^2\int\tfrac{1}{2}(1+\cos 2u)\mathrm{d}u \\ &= \tfrac{1}{2}a^2\left[u+\sin u\cos u\right],\end{aligned}$$

where we've used $\int\cos x\mathrm{d}x = \sin x$ from Table 1 (dividing by the constant $a = 2$ for twice the angle) and $\sin 2u = 2\sin u\cos u$ (from the formula above, for $\sin A+B$).

All you have to do now is put in the limits to get the definite integral.

At the lower limit $x = 0, \sin u = x/a = 0$: so $[...] = 0$.

At the upper limit $x = a$, $\sin u = (x/a) = 1$, $u = \frac{1}{2}\pi$: so $[...] = [\frac{1}{2}\pi + \sin\frac{1}{2}\pi\cos\frac{1}{2}\pi] = [\frac{1}{2}\pi + 1\times 0] = \frac{1}{2}\pi$.

Thus, $A = \frac{1}{2}a^2\left[u+\sin u\cos u\right]_{u=0}^{u=\pi/2} = a^2\pi^2/4$ and the area of the whole circle is four times this, namely πa^2. This is the result we already found in (4.19), but here we've had to work much harder to get it – because we kept on with rectangular Cartesian coordinates (x, y) instead of looking for a transformation that would make the integral easier to evaluate.

In this section we found a way of integrating by 'reversing' the rule for differentiating a 'function of a function'. Let's now look for an integration rule based on the recipe for differentiating a *product* of two functions

4.4 Integration 'by parts'

This time we start from the basic rule (2.21) for differentiating a product of two functions, $y(x) = u(x)v(x)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}, \tag{4.26}$$

where u and v are any two functions that we know how to differentiate. This result can be 'turned round', as usual, by noting that

$$\text{Given } \frac{\mathrm{d}y}{\mathrm{d}x} = f(x), \quad \text{then } y = \int f(x)\mathrm{d}x.$$

(Remember that differentiation and integration are simply inverse operations, D and D^{-1}, and that using the integral sign ($\int$) is just an alternative notation.) The 'turned round' version of (4.24) is thus

$$y = uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \tag{4.27}$$

and this means, rearranging the terms,

$$\int u\frac{dv}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{4.28}$$

If we can see that the integrand of an integral we can't do has the form $u(\mathrm{d}v/\mathrm{d}x)$, by suitably choosing the two functions u, v, then we can write it in terms of an integral of the product $v(\mathrm{d}u/\mathrm{d}x)$. And if we've made a good choice of u and v the new integral may be one that we *can* do.

Again, there's no general rule to tell you how to choose the two functions. You just have to use your imagination, trying out your guess to see if it makes things easier: sometimes it does and sometimes it doesn't. That's why you need lots of practice!

Here are a few examples to show how things work:

- (i) Suppose you want to evaluate $I = \int x \cos x \mathrm{d}x$. This can be looked at as the left-hand side of (4.25) if you choose

$$u = x, \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x.$$

In this case $v = \sin x$ and (4.25) becomes

$$\begin{aligned} \int x \frac{\mathrm{d}}{\mathrm{d}x} \sin x \mathrm{d}x &= x \sin x - \int \sin x \frac{\mathrm{d}}{\mathrm{d}x}(x) \mathrm{d}x \\ &= x \sin x - \int \sin x \mathrm{d}x \\ &= x \sin x + \cos x \end{aligned}$$

- (ii) A somewhat similar integral is $I = \int x \log x \mathrm{d}x$. This suggests you take

$$u = \log x, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1, \quad v = x,$$

so that (4.25) becomes

$$\begin{aligned} \int \log x \mathrm{d}x &= (\log x) x - \int x \frac{\mathrm{d}}{\mathrm{d}x} (\log x) \mathrm{d}x \\ &= x \log x - \int x \frac{1}{x} \mathrm{d}x = x \log x - x. \end{aligned}$$

(This is how we got the result given in Line 3 of Table 1.)

- (iii) Often the integrals we want to get come in pairs. For example,

$$\begin{aligned} P &= \int e^{ax}\cos(bx)\mathrm{d}x, \\ Q &= \int e^{ax}\sin(bx)\mathrm{d}x. \end{aligned}$$

To get P, try putting $\cos(bx) = u$ and the other factor $e^{ax} = \mathrm{d}v/\mathrm{d}x$, so that $v = e^{ax}/a$. Integrating by parts then gives, from (4.25),

$$\begin{aligned} P &= (e^{ax}/a)\cos(bx) - \int (e^{ax}/a) \times (-b\sin(\,bx)\mathrm{d}x \\ &= (e^{ax}/a)\cos(bx) + bQ/a. \end{aligned}$$

In the same way (do it yourself!) you'll find

$$Q = (e^{ax}/a)\sin(bx) - bP/a.$$

By rearranging these results we get a pair of simultaneous equations (Section 2.3 in Book 2):

$$aP - bQ = e^{ax}\cos\,bx, \qquad bP + aQ = e^{ax}\sin\,bx,$$

which can easily be solved to get P and Q separately. The result is (check it!)

$$\begin{aligned} P &= e^{ax}(b\sin\,bx + a\cos\,bx)/(a^2+b^2), \\ Q &= e^{ax}(a\sin\,bx - b\cos\,bx)/(a^2+b^2). \end{aligned}$$

There are many other special tricks for dealing with integrals which don't look do-able, but the examples above are enough to be going on with. In the Exercises at the end of the Chapter you'll find other integrals to try, with hints to help you get started.

4.5 When all else fails – do it with numbers!

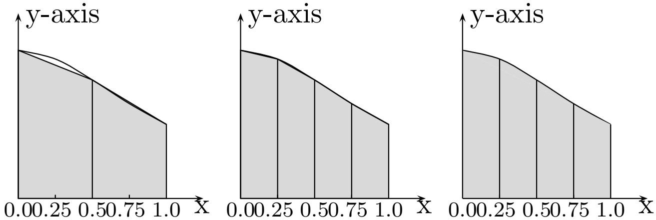
Very often, in applying the calculus, you'll be wanting to evaluate a definite integral; and you may not be able to find any way of getting an expression for the *indefinite* integral – so you can't just put in values of x at the upper and lower limits and take the difference. The only thing you can do in that case is to use *arithmetic* to get the area under the curve $y = f(x)$ between ordinates at $x = X_1$ and $x = X_2$: this is called **numerical integration**.

The simplest approximation is to divide the area into strips, all of width h say, going from X_1 to X_2 and add up the areas of all the strips. Let's get the area under the simple curve $y = f(x) = (1 + x^2)^{-1}$ between limits $X_1 = 0$ and $X_2 = 1.0$, which is given by the definite integral

$$A = \int_0^1 \frac{1}{1 + x^2}\mathrm{d}x$$

A very rough approximation is indicated in Fig.45(a), where we take just three points on the curve $y = f(x)$, with $x_1 =$

$X_1 = 0, x_3 = X_2 = 1.0$ and one point in between at $x_2 = 0.5$. The area is thus divided into two (rather wide) pieces with $h = 0.5$.



(a) $h = 0.5$ (b) $h = 0.25$ (c) Simpson

Figure 22

The ordinates at these values of x will have heights y_1, y_2, y_3 and the areas of the two pieces, formed by joining the points (x_1, y_1), (x_2, y_2), (x_3, y_3), will be $\frac{1}{2}(y_1+y_2)h$ and $\frac{1}{2}(y_2+y_3)h$, respectively. The first strip will have area $A_1 = \frac{1}{2}(y_1 + y_2)h$ and the second (really a triangle in this case) will have $A_2 = \frac{1}{2}(y_2+y_3)h$. A first approximation to the total area will thus be

$$A \approx A_1 + A_2 = (\tfrac{1}{2}y_1 + y_2 + \tfrac{1}{2}y_3)h.$$

In Fig.22(a) the width of each strip is $h = 0.5$ (i.e. unit width divided by two) and the total area comes out to be $A \approx 1.55h = 0.7750$. (Do the calculation for yourself!)

To get a better approximation, we can divide the whole area into n strips, using a smaller value of the width $h = 1/n$, instead of $h = 1/2$.

$$A \approx (\tfrac{1}{2}y_1 + y_2 + y_3 + \;....\; + \tfrac{1}{2}y_{n+1})h, \qquad (4.29)$$

where only the first and last y values have coefficient $\frac{1}{2}$. This is called the **trapezoidal rule** because each strip has the form known in geometry as a 'trapezium'.

Fig.22(b) shows the result of taking $n = 4$, which makes $h = 1/4$, only half as wide as the two strips in Fig.22(a). The ordinates at $x_1 = 0$, $x_2 = h$, $x_3 = 2h$, $x_4 = 3h$, $x_5 = 4h$ (remember $x = 0$ gives the *first* ordinate!) then come out as $y_1 = 1.0\, y_2 = 0.9412$, $y_3 = 0.8\, y_4 = 0.64\, y_5 = 0.5$; and if you repeat the calculation you'll find an approximate area $A \approx 3.1312h = 0.7828$.

The trapezoidal rule often gives poor results because the top of each strip is closed with a straight line, nothing like the curve we are trying to fill. It might be better to use a curved line and the simplest one we can think of has an equation of the second degree, $y = A + Bx + Cx^2$, which describes a parabola (Section 1.2 of Book 3). If we know three points on the curve we can choose the constants A, B, C so that the curve will pass through all of them. And if we do this for each double-strip in Fig.22(b) they will fit into the curve much better. Fig.22(c) shows the result of using two strips (with $h = 0.25$) to make each double-strip ($h = 0.5$), with a curved top. You choose the constants so that y_1, y_2, y_3 will define the first top. In the same way, y_2, y_3, y_4 will define the second top. Then you can use the new strips, with curved tops, in place of the two wide strips in Fig.22(a).

How does this help us to get a better approximation to the area under the curve? Well, we can get the area of a strip with a curved top by integration, which is easy for a curve like $y = A + Bx + Cx^2$. Then we can add the areas, just as we did for Fig.22(a), expecting them to give a better result.

To do this in detail, let's think of just one double-strip, using new names for the variables; we'll take $x = x_0$ for the central ordinate, with its top at $y = y_0$. The upper and lower boundaries of this piece will then be at $x = x_0 + h$ and $x = x_0 - h$, which we can call x_{+1} and x_{-1}, respectively. To make the arithmetic easy, we'll measure x from $x = x_0 = 0$ as origin. The three ordinates will then be at

$$x = x_{-1} = -h \qquad x_0 = 0 \qquad x_{+1} = +h$$

and will have heights

$$y = y_{-1} \qquad y_0 \qquad y_{+1}.$$

Putting $x = 0$ must give $y = A + B \times 0 + C \times 0^2 = y_0$, so the first constant will be $A = y_0$.

Putting $x = -h$ will give $y = A - Bh + Ch^2 = y_{-1}$, while with $x = +h$ we get $y = A + Bh + Ch^2 = y_{+1}$. From these two equations we can find the two unknowns, B and C.

Subtracting the first equation from the second, the A- and C-terms will cancel, leaving $2Bh = y_{+1} - y{-}1$. So the second constant must have the value $B = \frac{1}{2}(y_{+1} - y{-}1)/h$.

If instead we *add* the two equations, and put in the value already found $A = y_0$, we get $2y_0 + 2Ch^2 = y_{+1} + y_{-1}$. So C must have the value $C = \frac{1}{2}(y_{+1} + y_{-1} - 2y_0)/h^2$.

The next step is to find the area of the double-strip under the curve $y = A + Bx + Cx^2$. This is the integral $\int y\mathrm{d}x$

between limits at $x = -h$ and $x = +h$:

$$\int_{-h}^{+h} (A + Bx + Cx^2)\mathrm{d}x = \left[Ax + \tfrac{1}{2}Bx^2 + C(x^3/3)\right]_{-h}^{+h}.$$

When we put in the top and bottom limits and take the difference we'll get

$$Ah - A(-h) + C(h^3/3) - C(-h^3/3) = 2Ah + 2Ch^3/3$$

and on substituting the values of A and C this becomes (check this result carefully!)

$$\int_{-h}^{+h} y\mathrm{d}x = \frac{h}{3}\left[y_{-1} + 4y_0 + y_{+1}\right]. \tag{4.30}$$

Now we can get the area under the whole curve, between the limits at x_1 and x_n (the first and last ordinates). First put y_1, y_2, y_3 in place of y_{-1}, y_0, y_{+1} in the formula (4.30) to get the area of the first double-strip; then do the same for the next, using y_3, y_4, y_5; and so on until you get to the last ordinate y_n. When all these small areas are added together you'll have the whole area A in the form

$$\begin{aligned} A = [&(y_1 + 4y_2 + y_3) \\ &+(y_3 + 4y_4 + y_5) \\ &+(y_5 + 4y_6 + y_7) \\ &+ ....]\,(h/3) \\ = \;& [(y_1 + y_n) + 2(y_3 + y_5 + ...) + 4(y_2 + y_4 + ...)]\,(h/3) \end{aligned} \tag{4.31}$$

This result is called **Simpson's Rule**. It is easy to remember if you put it in words:

> Take the sum of the first and last ordinates, $y_1 + y_n$. Add *twice* the sum of the odd ordinates $(y_3, y_5, ...)$, lying in between them. And add *four times* the sum of the even ordinates $(y_4, y_6, ...)$. Then multiply the total by $h/3$.

The rule is easy to use and gives good results as long as the interval h is not too large and the integrand is well-behaved over the whole range of integration.

An example is shown in Fig.22(c), using the same curve as in (a) and (b), with $h = 0.25$. The five ordinates needed are easily found; and using them gives a much better approximation to the integral. According to Table 2, the indefinite integral of $y = 1/(1 + x^2)$ is $\tan^{-1} x$, the angle in radians whose tangent is x: at the upper limit of the definite integral, $x = 1$ and this is the tangent of the angle $\pi/4$ or 45 degrees, while at the lower limit, $x = 0$ and is the tangent of the angle zero. The difference of the two is the value of the definite integral we were calculating – and it should therefore be $\pi/4$. Since $h = 1/4$ our approximate area is 3.1416/4, corresponding to $\pi \approx 3.1416$. The approximation is good to four figures after the point, even though we divided the area into only four strips and worked only to four decimal places. If you use a pocket calculator and carry more figures you can easily get a better value $\pi \approx 3.141593$ by using ten strips instead of only four.

We've been talking about ways of getting definite *integrals* by numerical methods but you can also *differentiate* a function in similar ways. In finding Simpson's rule, for example, the function we wanted to integrate was represented, piece-by-piece, by fitting its graph to a polynomial $y = A + Bx + Cx^2$; and by doing the integration we got the area of each piece. We can just as easily differentiate the polynomial to get the *slope* of the curve, for any value of x. So let's note, in passing, a result similar to (4.30) but giving the value of $\mathrm{d}y/\mathrm{d}x$ at a middle point $x = x_0$ instead of the area of the strip between ordinates at $x_{-1} = x_0 - h$ and $x_{+1} = x_0 + h$. The simplest approximation (draw a graph to see what it means!) is

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{1}{2h}(y_{+1} - y_{-1}). \tag{4.32}$$

That was with only three ordinates. If you want a much better result you can use five instead, fitting the curve with $y = A+Bx+Cx^2+Dx^3+Ex^4$, finding the constants in terms of ordinates at x_0, $x_0 \pm h$, $x_0 \pm 2h$ and then differentiating. The result is surprisingly simple:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{1}{12h}\left[(y_{-2} - y_{+2}) + 8(y_{+1} - y_{-1})\right]. \tag{4.33}$$

If you ever get completely stuck in trying to solve a problem, don't forget that you can always fall back on simple arithmetic – after all that's where mathematics first started! Nowadays there are whole books on numerical methods and even the smallest computers can do all the arithmetic for you.

Exercises

1) Verify that $\mathsf{D}\mathsf{D}^{-1} = \mathsf{D}^{-1}\mathsf{D} = \mathsf{I}$ when the *operators* $\mathsf{D}, \mathsf{D}^{-1}$ are applied to the functions listed in Table 1. (*Hint*: First work through the example given in the text (pages 94-94), where $f(x) = x^n$. Then try your hand at some of the other functions, making similar steps.)

2) Find $\mathrm{d}y/\mathrm{d}x$ for the functions $y = f(x)$ listed in Table 2 and again verify that $\mathsf{D}\mathsf{D}^{-1} = \mathsf{D}^{-1}\mathsf{D} = \mathsf{I}$.

3) Think of "multiply by x" as an *operator* x and verify that $\mathsf{Dx} - \mathsf{xDx}$ is equivalent to the identity operator, I, when applied to any well-behaved function $f(x)$. (This is true for all values of x in the whole interval $(-\infty, +\infty)$.) (*Hint*: Keep the function $f(x)$ in there, for the operators to work on, dropping it only at the end when you've found the result.)

4) Use equation (4.13) to get an expression for the distance s_{12} between points (x_1, y_1) and $x_2, y_2)$ on a parabolic curve where $y = x^2$. Try to find some way of evaluating the integral, using the methods of Section 4.3. If you can't, then use the methods of Section 4.5 to get a good approximation.

5) Work through the calculation, starting on p.111, of the volume of water in a shallow pond. But now suppose you're thinking of water (or wine) contained in a deep 'goblet' of parabolic form $z = A + Br + Cr^2$, z being the level of the liquid relative to the base (where $z = 0$) and r the radius of the container at level z.

Use the method explained in the text to choose the constants A, B, C for a goblet of depth $Z = 6$ cm and width $R = 2.5$ cm at the brim.

When the goblet is full to the brim how much wine will it hold? And when it looks to be still roughly half full ($z = \frac{1}{2}Z$) how much is left?

6) Evaluate the following indefinite integrals (in which a, b are constants):

(a)

$$\int \frac{cx}{ax^2 + b} \mathrm{d}x$$

(*Hint*: use (4.25))

(b)

$$\int x \cos x^2 \mathrm{d}x$$

(*Hint*: put $x^2 = u$)

(c)

$$\int \frac{\sqrt{x}}{\sqrt{x} - 1} \mathrm{d}x$$

(*Hint*: put $\sqrt{x} - 1 = t^2$)

(d)

$$\int x^3 \sqrt{x^2 - 1} \mathrm{d}x$$

(*Hint*: put $x^2 - 1 = t^2$)

(e)

$$\int x^2 e^x \mathrm{d}x$$

(*Hint*: use (4.28), putting $e^x = \frac{\mathrm{d}v}{\mathrm{d}x}$)

7) Show how the results in the last Exercise will be changed if you replace x by cx ($c =$ constant).

Chapter 5

Power series, convergence, and Taylor's theorem

5.1 Sequences, series and sums

Even in Book 1 (Section 5.1) we came across sets of numbers with very special (and useful!) properties. For example, the "exponential series"

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + ... \qquad (5.1)$$

is the **sum** of a **sequence** of terms 1, 1, 1/2, 1/6. 1/120, ... which goes on forever. The general member of the sequence has the form $1/n!$, where $n! = 1 \times 2 \times 3 \times 4 \ ... \ \times n$ is the product of the first n natural numbers, known as "factorial n", with $0! = 1$.

Similarly, the sequence with terms $a, ax, ax^2, ax^3,\ ...$ forms a "geometric progression". The sum of the terms in the sequence, namely

$$S = a + ax + ax^2 + ax^3 + \ ... \ , \tag{5.2}$$

forms a **geometric series** in which every term, u_r say, depends on the two numbers a and x. Thus

$$S = u_1 + u_2 + u_3 + u_4 + \ ... \quad (u_r = ax^{r-1}). \tag{5.3}$$

Now it's time to take these ideas a bit further.

In general, a sequence is any set of terms which can be arranged in a definite order, such as $u_1, u_2, u_3,\ ...$, where the subscript shows the 'ordinal number' of the term (first, second, third, etc.). This sequence becomes **finite** when there is a last term, u_n, for finite n. When there is no last term, the terms going on 'forever', the sequence is **infinite**.

Very often the general term u_r in a sequence is obtained from its ordinal number r by some simple rule, as in the geometric progression, where the rth term is $u_r = ax^{r-1}$. In the exponential series (5.1) the general term is $u_r = 1/r!$ and the ordinal number r includes the case $r = 0$, which gives the first term on the right in (5.1) – provided we *define* $0! = 1$. (Of course a product of *no* integers looks like nonsense; but, by agreeing to write $r! = 1$ in the special case where $r = 0$, the first term in the sequence falls into line with all the others, $u_0 = 1/0! = 1$. Then $r = 1$ gives the next term, $u_1 = 1/1! = 1$, and after that everything is fine.)

When a sequence is infinite it may have a **limit**, namely the value which the 'last' term we look at (u_r, say) *approaches* as we take r bigger and bigger. This limit is written $\lim_{r\to\infty}(u_r)$; and when this limit is a finite number and is unique (we get the same value in whatever way we find it!) we say the sequence **converges**. In all other cases, the sequence is said to **diverge**: there will be no finite limit, or the last term may 'jump about' (depending strongly on the value of r. We'll nearly always be talking about *convergent* sequences.

When the terms in a sequence are *summed*, as in (5.1) and (5.2), the result is called a **series**. It is also useful to think about **partial sums** in which only the first n terms are included. The partial sum, or the 'sum to n terms', is thus defined by

$$S_n = u_1 + u_2 + u_3 + u_4 + \ \dots\ u_n = \sum_{r=1}^{n} u_r. \qquad (5.4)$$

Just as an infinite sequence may either converge or diverge, the *series* (i.e. the sum of all the terms in the sequence) may converge or diverge. The series **converges** when the partial sum (5.4) approaches a unique and finite limit S, as n becomes bigger and bigger. And we write this as

$$S = \lim_{n\to\infty} S_n. \qquad (5.5)$$

Otherwise the series **diverges**.

The series (5.1) is convergent. But the series (5.2), which depends on the variable x, is convergent only when the magnitude of x is less than 1 i.e. when $|x| < 1$. Thus, for $x \geq 1$ the value of S_n increases without limit as $n \to \infty$ and the series therefore diverges.

In Book 1, where we first met series in Section 5.1, we just supposed that a series would converge when the terms got smaller and smaller; but now let's try to be more precise. We ask the general question:

How can we tell if a given series converges?

It is certainly *necessary* that the sequence of terms in (5.4) converges – that as n becomes indefinitely large then $u_n \to 0$. For this means that on adding further terms the partial sum S_n will not change any more. But this is not enough: it is not **sufficient** to guarantee that the sum of *all* the terms up to u_n will converge to a finite result. For example, look at the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ ... \ + \frac{1}{\sqrt{n}} + \ ... \ .$$

The nth term, $1/\sqrt{n}$ does go to zero for $n \to \infty$; but S_n *does not.* To see that this is true, note that all the terms that come before u_n are greater than u_n, so adding all the first n terms together will certainly give you something bigger than n times the last one: $S_n > n \times (1/\sqrt{n})$. In other words, $S_n > \sqrt{n}$. As n becomes indefinitely large, so does S_n and the sseries therefore diverges.

There are various conditions which will guarantee the convergence of a given series, being both **necessary and suf-**

ficient. Some of these **convergence tests** are based on making a comparison with a series that is known to converge (e.g. the geometric series, whose partial sum S_n was found in Book 1, equation (5.1), for any value of n): if the terms in the given series are all smaller than the corresponding terms in a convergent geometric series, then the given series must also converge. Other tests are based on comparing the nth term of the given series with the one that follows it, to see whether the terms are getting bigger or smaller. We'll use only one of these **ratio tests**, that due to the French mathematician d'Alembert (1717- 1783):

$$\text{If} \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \quad \text{the series converges.} \tag{5.6}$$

Otherwise the series diverges (or, if the ratio is 1, may need further testing).

Let's test two of the series we've used already:

Example 1. The geometric series.

Suppose we didn't know the sum of the geometric series $1 + x + x^2 + x^3 + \ ...$ (which is (5.2) with $a = 1$), but only the general term $u_n = x^n$. The limit in (5.6) is then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x| \tag{5.7}$$

and the series converges as long as $|x| < 1$. For $|x| = 1$, ($x = \pm 1$) the test doesn't say anything: but an infinite sum of 1s is clearly infinite and the series diverges.

Example 2. The exponential function.

The function $y = e^x = \exp x$ is defined by the series

$$y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \; ... \tag{5.8}$$

and reduces to (5.1) on putting $x = 1$. The general term is thus $x^n/n!$ and the ratio in (5.6) becomes

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} = \frac{x}{n+1}. \tag{5.9}$$

On going to the limit $n \to \infty$, we see

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 \tag{5.10}$$

and the series therefore converges for *all* values of the variable x. Thus, $y = \exp x$ is a unique function of the real variable x, for all values of x in the interval $(-\infty, +\infty)$. When we first studied some of the remarkable properties of this function (in Chapter 4 of Book 2) we got them mainly by using simple examples and guesswork: we didn't even know if we could use e^x as if it were a single number, because we didn't know much about *convergence* and *limits*. We just supposed that e^x and e^y could be multiplied together by multiplying every term of one series by every term of the other and adding the results; and it looked as if the answer should be the function e^{x+y}, obtained from the same series but with the variable $x + y$ in place of x or y. But no real mathematician would dare to do a thing like that without first proving that every step made sense! Now we've taken the first step, by showing that the exponential series has a limit for every value of the number x and that this limit is a unique finite number. The next step (which we're not going to take!) is to show that when we multiply two convergent series the result will also be a convergent series, whose limit will be the product of the two separate limits. For now, we'll just assume that these 'common sense' ideas are correct.

5.2 Power series and their convergence

Functions like (5.2) and (5.8) are examples of **power series**, consisting of ordered sums of *powers* of the independent variable x, each with a constant coefficient: in general

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ \dots \ , \qquad (5.11)$$

in which $a_0, \ a_1. \ a_2 \ \dots$ are numerical constants, is a power series representing the function $f(x)$. In this case the powers are positive integers, including zero (which gives the leading term $a_0 = a_0 x^0$).

The condition for convergence (5.6) becomes

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} |x| \left| \frac{a_{n+1}}{a_n} \right| < 1, \qquad (5.12)$$

which can also be written

$$|x| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R \qquad (5.13)$$

Here R, the limit standing on the right in the last equation, is called the **radius of convergence** of the series. If you draw a circle of radius R around the point with $x = 0$ on the x-axis, then the series will converge only for x values lying within the circle (i.e. between $-R$ and $+R$.

You'll be wondering why we're talking about the radius of a circle! It's because mathematical analysis is a big subject and it deals with all kinds of numbers, including the *complex numbers* (Book 1, Chapter 5) such as $z = x + iy$, where i is the 'imaginary unit' with the property $i^2 = -1$. Complex numbers like $x + iy$, depend on *pairs* of 'ordinary' (real) numbers, x and y, and have a 'magnitude' $|z|$ which is also real and is given by $|z|^2 = x^2 + y^2$. So, if you think of a number pair as the coordinates of a point in a plane, then $|z|$ will be its distance from the origin ($x = y = 0$); and the condition $|z| < R$ will hold for all points inside the circle of radius R – not only for the *real* numbers represented by points on the x-axis between $-R$ and $+R$. But analysis, applied to functions of a complex variable will need a whole book to itself!

For the geometric series (5.2), $R = 1$; but for the exponential series (5.8), $R = \infty$. The exponential series $y = \exp z$ is in fact convergent for *all* values of the variable z, real or complex. This means that the other series, for functions such as $y = \sin x, y = \cos x$, already used in Book 1, will also be convergent for all values of the variable x, since they are simply combinations of exponential functions $\exp ix$ and $\exp -ix$, namely e^z for two imaginary values $z = \pm ix$ of the independent variable. In the next Section we start to look at the general question of how to represent any given function as a power series.

5.3 Taylor's Theorem

At the end of the last Chapter (in Section 4.5) we used a three-term power series $y = f(x) \approx A + Bx + Cx^2$ to represent any given function $y = f(x)$ in the interval from $(x_0 - h)$ to $(x_0 + h)$, x_0 being the mid-point of the interval. In that way we were able to get the area of a double-strip, of width $2h$, under the curve, in terms of the corresponding ordinates (function values) $y_{-1} = f(x_0 - h)$, $y_0 = f(x_0)$, $y_{+1} = f(x_0 + h)$. And by adding the areas of many such strips we could estimate the value of the *definite integral*, represented by the area under the whole curve between any given limits $x = a$ and $x = b$. We also noted that using a five-term approximation $y = A + Bx + Cx^2 + Dx^3 + Ex^4$ would give a better result.

Let's now take a more general case, using a polynomial of the nth degree,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots + a_n x^n, \tag{5.14}$$

in which there are $(n+1)$ constant coefficients $a_0, a_1, \ldots a_n$. The coefficients could be found, again, in terms of the ordinates $y_0, y_1, \ldots y_n$ corresponding to a set of x-values $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots x_n = x_0 + nh$ (using h for the common spacing between one ordinate and the next).

But instead of finding the coefficients in that way, which is quite difficult, we'll use a beautiful trick discovered by a student of Newton, called Taylor, He noticed that every one of them could be found by *differentiating* the function $f(x)$ and then putting $x = 0$. Thus, a_0 is the only terms left when $x = 0$ – for all the powers of x are then zero and we are left with $f(0) = a_0 + a_1 \times 0 + a_2 \times 0 + \ ... \ $ – and so $a_0 = f(0)$.

Next we differentiate and use the usual shorthand notation for the results:

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}f}{\mathrm{d}x} = 0 + a_1 + 2xa_2 + 3x^2a_3 + 4x^3a_4 + \ .. \\
f''(x) &= \frac{\mathrm{d}^2f}{\mathrm{d}x^2} = 0 + 0 + 2a_2 + 6xa_3 + 12x^2a_4 + \ ... \\
f'''(x) &= \frac{\mathrm{d}^3f}{\mathrm{d}x^3} = 0 + 0 + 0 + 6a_3 + 24xa_4 + \ ... \\
f^{(4)}(x) &= \mathsf{D}^4f(x) = 0 + 0 + 0 + 0 + 24a_4 + \ ... \ ,
\end{aligned}$$

and so on. (Note that the raised integer in $f^{(4)}(x) = \mathsf{D}^4f(x)$ is used to mean "$f(x)$ differentiated 4 times" and that the symbol D, already used in earlier Chapters (e.g. in equation (4.2)) stands for the *operator* $(\mathrm{d}/\mathrm{d}x)$ – here applied 4 times.)

Finally, let's put $x = 0$ in all the above equations. The results are

$$f(0) = a_0, \ f'(0) = a_1, \ f''(0) = 2a_2, \ f'''(0) = 6a_3, \\ ... \ , f^{(m)}(0) = m!a_m, \ ... \ , \qquad (5.15)$$

where the general term (shown last) contains the *factorial* $m! = 1.2.3. \ ... \ m$. Using (5.15) in (5.14) we obtain **Taylor's**

Theorem for a finite polynomial of the nth degree:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ ..$$
$$+\frac{x^n}{n!}f^{(n)}(0). \qquad (5.16)$$

Note that the series ends after $n+1$ terms, derivatives higher than that with $m = n$ all being zero; and that the expansion of $f(x)$ is around the origin, $x = 0$, not around a point in the middle of the range over which the series may be needed. Also the result has not been proved for *any* function – only for a polynomial. But it's a good start!

A more general form of the theorem, in which the function $f(x)$ is expanded about any point, not just about $x = 0$, can be found as follows. We introduce a new variable $\bar{x} = x + h$, so that when $x = 0$ $\bar{x}$ will take the value $\bar{x} = h$, and then look at the function $f(\bar{x}) = f(x + h)$. Keeping x constant, we can then move along the curve $y = f(\bar{x})$ by changing h and thinking of $y = f(x + h)$ as a new function $y = g(h)$. Since $\bar{x}$ and h differ only by a constant (x), the derivatives $dy/d\bar{x}$ and dy/dh, will be equal; and this will be true also on repeating the differentiation. Thus

$$g(h) = f(\bar{x}),\ g'(h) = f'(\bar{x}),\ g''(h) = f''(\bar{x}),\ \text{etc.}$$

Now let's use (5.16) on the new function $g(h)$, getting

$$g(h) = g(0) + hg'(0) + \frac{h^2}{2!}g''(0) + \frac{x^3}{3!}g'''(0) + \ \text{etc}$$

and finally put this in terms of $f(x+h) = g(h)$ and its derivatives, noting that $h = 0$ corresponds to $\bar{x} = x$. The result is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x), \qquad (5.17)$$

where the function and its derivatives, on the right, are all evaluated at a general point x, rather than at the origin $x = 0$. In many applications, this is a more useful form of Taylor's expansion than the special case (5.16).

As long as we consider only the expansion of a *finite polynomial*, the above results are generally valid; but wouldn't it be nice if they could be used for *any* kind of function that could be differentiated! It seems likely that this will be so, because we already know that a given function can usually be fitted fairly accurately, at least over a short range, by a polynomial with only a few terms; and by taking more and more terms one could hope to get an almost exact representation over the whole range in which the function was 'well-behaved'. But to *prove* that this can be done is difficult: it requires us to talk about the **remainder** R_n – the sum of the remaining terms when we stop after the first n – and that is stuff for real mathematicians. Usually we'll just assume that Taylor expansions can be found for all differentiable functions. The examples that follow will show how this can be done

Some examples of Taylor expansions

Although we aleady know some series representing common functions, such as $\exp x, \sin x, \cos x$, it's interesting to see how they also follow from Taylor's theorem, provided we know how to differentiate the functions. Let's look at one or two of them and then take a new one.

Example 1. The exponential and logarithmic series

Suppose we didn't have a series for the function e^x, but only knew that it was a function whose first derivative $f'(x)$ gave us back the function itself: $f'(x) = f(x)$. In that case, by differentiating again, and again, we can find *all* the derivatives. Thus,

$$\begin{aligned} \mathsf{D}f &= \frac{\mathrm{d}f}{\mathrm{d}x} = f'(x), \\ \mathsf{D}^2 f &= \mathsf{DD}f = f''(x), \\ \mathsf{D}^3 f &= \mathsf{DD}^2 f = f'''(x), \ \ldots\,, \end{aligned} \tag{5.18}$$

and so on. It follows at once, using (5.16) that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ \ldots\,. \tag{5.19}$$

This agrees with the series given in (3.22); but here we have found it by solving the differential equation $f'(x) = f(x)$, first stated in (3.24).

When we first defined the logarithmic function $y = \log x$, we didn't find a series for it: instead we had to express it as an *integral*, $\log x = \int (1/x)\mathrm{d}x$. Now we'll show how it can also be written as a Taylor expansion.

Since $1/x$ is a 'nasty' function, which 'blows up', along with all its derivatives, at the point $x = 0$, let's change to a new variable t by using $1 + t$ in place of x in the integral form. Thus

$$\log x = \int \frac{1}{1+t}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \int \frac{1}{1+t}\mathrm{d}t = \log(1+t).$$

If now we take the *definite* integral, by putting in limits $t = 0$ (lower) and $t = x$ (upper), we find

$$\int_0^x \frac{1}{1+t}dt = [\log(1+t)]_{t=0}^{t=x} = \log(1+x), \qquad (5.20)$$

since the lower limit is $\log 1 = 0$.

We can now get the series we want by noting that $(1+t)^{-1}$ is the sum to infinity of a simple geometric progression (Book 1, Section 5.1):

$$(1+r)^{-1} = 1 + r + r^2 + r^3 + \ \dots\ ,$$

with the 'common ratio' r put equal to $-t$. On using this result as the integrand on the left in (5.20) and integrating term by term it follows that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \dots\ . \qquad (5.21)$$

It is important to note that the series converges only for values of x in the range -1 to $+1$.

Example 2. The sine and cosine series

The functions $\sin x$ and $\cos x$ are *periodic*, their values repeating whenever the variable x increases by 2π or. with a change of sign, by π. Some of their properties are summarized in Book 2, Chapter 4; and we already know from (3.7) that, using the D-notation as in (5.19),

$$\mathsf{D}\sin x = \cos x, \qquad \mathsf{D}\cos x = -\sin x. \tag{5.22}$$

From these properties we get, by repetition,

$$\begin{aligned} \mathsf{D}^2\sin x &= \mathsf{D}\cos x = -\sin x, \\ \mathsf{D}^3\sin x &= \mathsf{D}(-\sin x) = -\cos x, \\ \mathsf{D}^4\sin x &= \mathsf{D}(-\cos x) = \sin x, \end{aligned}$$

etc., where each term follows at once from the one before it.

To get the Taylor expansion of $\sin x$, all these derivatives must be evaluated – but only at $x = 0$; and that is easy! The sine terms are all zero, while the cosine terms are all ± 1, the sign changing in going from one non-zero term to the next. On putting the results in (5.16) we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ \dots\ . \tag{5.23}$$

The Taylor series for the cosine follows in the same way, starting from $\mathsf{D}\cos x = -\sin x$ and finding

$$\begin{aligned} \mathsf{D}^2\cos x &= -\mathsf{D}\sin x = -\cos x, \\ \mathsf{D}^3\cos x &= \mathsf{D}(-\cos x) = +\sin x, \\ \mathsf{D}^4\cos x &= \mathsf{D}(\sin x) = \cos x, \end{aligned}$$

etc., and when $x = 0$ the derivatives again take only the values $0, \pm 1$. Substitution in (5.16) then gives

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ \dots \ . \qquad (5.24)$$

Like the exponential series, these expansions converge for all values of x in the infinite interval $(-\infty, +\infty)$.

Example 3. The binomial series

We met the **binomial series** in Section 3.1, where we expanded $(a+b)^n$ and found the first three terms of the series in (3.2). If we put $a = 1, b = x$ the series becomes

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ \dots \ . \qquad (5.25)$$

The result was used in differentiating the function $y = x^n$, where n was a positive integer, but now we want to show that the same series holds good for all real values of n, positive or negative, rational or irrational. Taylor's expansion (5.16) allows us to do that.

Here the function we're expanding is $f(x) = (1+x)^\alpha$, where both x and α are now any real numbers. And we'll need all the derivatives, $f'(x) = \mathsf{D}f(x)$, $f''(x) = \mathsf{D}^2 f(x)$, ... , even though we don't yet know how to differentiate $(1+x)^\alpha$ for general values of α. To start, let's just suppose we can use the same rule as for when α is an integer: in that case $\mathsf{D}(1+x)^\alpha = \alpha(1+x)^{\alpha-1}$ (in words "multiply by the exponent α and then reduce α, in the exponent, by 1").

Using this rule, the derivatives we need will be

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1},\ f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}, \\ f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}\ \ldots, \end{aligned}$$

and so on. And when we put $x = 0$ the results are

$$f'(0) = \alpha,\ f''(0) = \alpha(\alpha-1),\ f'''(0) = \alpha(\alpha-1)(\alpha-2),$$

and soon. From (5.16) it then follows that the first few terms of the Taylor expansion are

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ .. \tag{5.26}$$

This is exactly the same as (5.25), except that instead of the integer n we are now supposing α is *any* real number.

Of course we should really *prove* that the differentiations can be done just as we did in differentiating x^n in Section 3.1. This is a bit tough, so we'll put it in small print and you can come back to it later.

Note: how to differentiate x^{α} for any α

Look back at the section starting with equation (3.26). This equation defines the logarithmic function as the *inverse* of the exponential: in general, if $p = e^q$ we can turn it round and say $q = \log p$ (it is the power to which you must raise the *base* e to get back p). So we can also write

$$p = e^q = e^{\log p},\ \text{for any number } p.$$

This is an *identity*: the right-hand side of the equation $p = e^{\log p}$ is just a different way of writing the same thing,p.

Here we have to deal with $y = x^\alpha$; and it helps (you'll see why in a minute) to bring in the logarithm by writing $x = e^{\log x}$. For then, using what we know from (3.23), we can say

$$y = x^\alpha = e^{\alpha \log x}$$

and we can differentiate this by 'changing the variable' according to (2.22). To do this, we put $\alpha \log x = u$, a new variable, and then use $y = e^u$.. We know that $(\mathrm{d}y/\mathrm{d}u) = y = e^u$ (the basic property of the exponential function); and we know from (2.22) that $(\mathrm{d}y/\mathrm{d}x) = (\mathrm{d}y/\mathrm{d}u)(\mathrm{d}u/\mathrm{d}x)$. So now we can say

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{\alpha \log x} \frac{\mathrm{d}}{\mathrm{d}x}(\alpha \log x) = x^\alpha \frac{\mathrm{d}}{\mathrm{d}x}(\alpha \log x) = x^\alpha (\alpha \frac{1}{x}) = \alpha x^{\alpha - 1}$$

where we've used the property (3.28) of the logarithmic function. The final result is thus

$$y = x^\alpha : \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \alpha x^{\alpha - 1} \qquad (5.27)$$

– just as if α were an integer.

You can find a Taylor expansion of *anything* about any point where the derivatives all exist – given the patience to do all the differentiations!

Exercises

1) Obtain power series for the following functions around the point $x = 0$:

(a) $\sin x/x$, (b) $\cos x/x$,
(c) $(1 - \cos x)/x^2$, (d) $(\sin x - x)/x^3$.

What are the limiting values of the functions (if they exist) for $x \to 0$?

(*Hint*: Use the series in (5.23) and (5.24))

2) Derive the expansion

$$\tan x = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \ ... \ ,$$

by evaluating the first few derivatives in a Taylor series. (*Hint*: You need to differentiate $\tan x$ several times, putting $x = 0$ in the results; and don't forget that $\sec^2 x = 1 + \tan^2 x$.) Can you show that the next term will be $272x^7/7!$?

3) Use the same method to obtain the following expansions:

(a)

$$e^x \cos x = 1 + x - \frac{2}{3!}x^3 - \frac{2^2}{4!}x^4 - \frac{2^2}{5!}x^5 + ...$$

(b)

$$e^x \sin x = x + x^2 + \frac{2}{3!}x^3 - \frac{2^2}{5!}x^5 - \frac{2^2}{6!}x^6 + ...$$

Sketch the functions for some small values of x and explain why the series contain both odd and even powers of the variable, while the series in (5.23) and (5.24) do not.

4) From the expansion (5.26) of $(1 + x)^\alpha$, obtain the series

$$\begin{aligned}(x+y)^\alpha &= x^\alpha + \alpha x^{\alpha-1}y + \frac{\alpha(\alpha-1)}{2!}x^{\alpha-2}y^2 \\ &\quad + \frac{\alpha(\alpha-1)(\alpha-3)}{3!}x^{\alpha-3}y^3 + \ ...\end{aligned}$$

and show that it converges for $y/x < 1$ for all real values of α.

Chapter 6

A quick look at some things you'll need later

6.1 Functions of more than one variable

At the beginning of this book, in Section 3.1, we noted that a function may depend on more than one variable: in climbing a hill we might go a distance x towards the East, and then a distance y towards the North (both measured relative to an x-axis and a y-axis, starting at the origin O in Fig.23). At the end, we'll be at a some height z above the horizontal plane which contains the x- and y-axes: we'll have moved a distance z along the vertical direction – indicated by the z-axis.

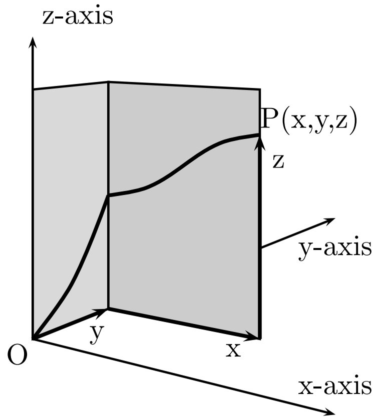


Figure 23

(If you're not sure about what all this means, look back at Book 2!)

The final point P which you reach has *coordinates* x, y, z and is referred to as "the point P(x, y, z)". And if you hold z fixed and keep walking, going neither higher nor lower, then your path will be a **contour line**: it will be described by a relationship $f(x, y) = z =$ constant. There will be a contour line for any value of the height, as long as you stay on the surface, giving a whole 'family' of contours, $z_1 = f_1(x, y), z_2 = f_2(x, y)$, ... *etc.*. (If you've ever studied a map you will know that families of contour lines can give a clear picture of the form of the land you're walking over. For example, near the top of a hill the contours may be roughly circular, with the top at P being near the middle).

In this example $z = f(x, y)$ expresses the height of any point on the surface as a function of two independent variables, the distances moved along the x and y directions, but in Science we're always meeting relationships of this kind in which the variables have nothing to do with distances. So we need to extend what we know about functions of one variable to functions of two, or many, variables. It's easy to do this for two variables, because we can get help from pictures, but once we've learnt how to express the pictures in mathematics we can do the same thing for any number of variables: so let's start from Fig.23.

In going along the y direction, we kept x fixed with the value $x = 0$ so that only y and z were changing as we went up the steep slope, with $z = f(0, y)$ – a function of one variable only. After going in the y direction (North, say) until $y = 1$, we turn and go towards the East. The path for this direction will now be described by $z = f(x, 1)$, again a function of one variable only – but a *different* one.

Now in the Calculus we're interested in what happens when the variables change only by very small amounts, $\delta x, \delta y, \delta z$, and in ratios such as $\delta z/\delta x$, which measure *rates of change*, or *slopes*. If we're at the top of the hill in Fig.23 and go a little bit (δx) further in the x direction, then the change in z will be

$$\delta z \approx \frac{\mathrm{d}}{\mathrm{d}x} f(x, y)_{\text{y fixed}} \times \delta x;$$

but if we went in the y direction, by δy, it would be

$$\delta z \approx \frac{\mathrm{d}}{\mathrm{d}y} f(x, y)_{\text{x fixed}} \times \delta y.$$

It's convenient to rewrite these two changes using **differentials** instead of 'deltas' (look back at Chapter 2 for a reminder of what they are): with this notation, the first change becomes

$$\mathrm{d}z = \left(\frac{\partial f}{\partial x}\right)_y \mathrm{d}x$$

and the second becomes

$$\mathrm{d}z = \left(\frac{\partial f}{\partial y}\right)_x \mathrm{d}y.$$

When both changes are made together the total change will then be

$$\mathrm{d}z = \left(\frac{\partial f}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_x \mathrm{d}y. \tag{6.1}$$

(Note that, just as we've often used $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}f/\mathrm{d}x$ to mean the same thing when $y = f(x)$, we count $\partial z/\partial x$ and $\partial f/\partial x$ as the same when $z = f(x, y)$; it doesn't matter if we use the same name for the quantity z or the function $f(x, y)$, which tells us how to get it from x and y)

The quantities in big round brackets, containing 'curly' d's, are called **partial derivatives**. They are just like ordinary derivatives except that they are defined for functions of more than one independent variable and the subscript shows any variable that is *held fixed* (i.e. treated like a constant). Thus, for a function $f(x, y)$ there are two partial derivatives $(\partial f/\partial x)_y$ and $(\partial f/\partial y)_x$. They are defined (without the pictures!) by

$$\left(\frac{\partial f}{\partial x}\right)_y = \lim_{\delta x \to 0}\left(\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right),$$
$$\left(\frac{\partial f}{\partial y}\right)_x = \lim_{\delta y \to 0}\left(\frac{f(x, y+\delta y) - f(x,y)}{\delta y}\right) \tag{6.2}$$

– exactly as we defined ordinary derivative in Chapter 2, Section 2.3, which you should read again, just to make sure.

Example

As an example of how things go, you can look at the function

$$z = f(x,y) = x^2 - 2xy - 3y^2$$

and find the two first derivatives, one for each of the two variables. Thus

$$\left(\frac{\partial f}{\partial x}\right)_y = 2x - 2y, \quad \left(\frac{\partial f}{\partial y}\right)_x = -2x - 6y.$$

It's easy; all you have to do is differentiate with respect to one variable, treating the other just as if it were a constant. And with only *two* independent variables you can even drop the subscripts, which only mean "the other one". That's what we'll do from now on.

Now try some of the Exercises at the end of the Chapter.

Towards the end of Section 2.3, in talking about functions of only one variable, we were able to define 'higher' derivatives by differentiating the 'first' derivative to get 'second' and 'third' derivatives: $\mathrm{d}^2 f/\mathrm{d}x^2$ and $\mathrm{d}^3 f/\mathrm{d}x^3$, etc.; and we can do the same for functions of more than one variable. Thus, if $z = f(x, y)$ we can find first derivatives, such as $(\partial f/\partial x)$ in (6.2), and then go on to get

$$\left(\frac{\partial^2 f}{\partial x^2}\right) = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial f}{\partial x}\right), \qquad \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right), \tag{6.3}$$

and so on.

We have to take care with the order of the differentiations: usually, in

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right),$$

the operation nearest the function f is the one you do first – but some books use the opposite rule, so watch out!. The rule used here is close to the one in which the operators $\mathsf{D}_x, \mathsf{D}_y$ are used. The second derivative we're talking about can be written in this way as $\mathsf{D}_x\mathsf{D}_y\, f$, which is simpler and clearer. And, just as with functions of one variable, we can make things even easier: the notations for first derivatives are

$$\left(\frac{\partial f}{\partial x}\right) = \mathsf{D}_x\, f = f_x, \quad \left(\frac{\partial f}{\partial y}\right) = \mathsf{D}_y\, f = f_y, \tag{6.4}$$

while for the second derivatives we have

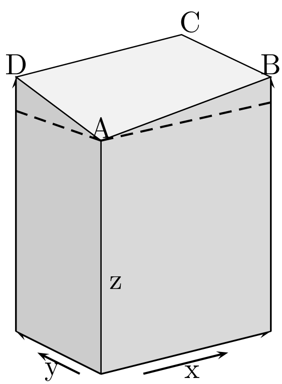
$$\left(\frac{\partial^2 f}{\partial x^2}\right) = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial f}{\partial x}\right) = \mathsf{D}_x\mathsf{D}_x\, f = f_{xx} \tag{6.5}$$

and

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \mathsf{D}_x\mathsf{D}_y\, f = f_{xy}. \tag{6.6}$$

(Notice that we don't need to put primes on the derivatives, one each time we differentiate, because the subscripts on f_x, f_{xy}, in (6.5) and (6.6), tell us which differentiations we've done and how many.) The older books use the full notation, on the left in these equations, which goes back to Leibnitz and the early days of the Calculus, but we'll often use the simpler forms shown on the right. So be sure you know about them all.

You can see what the derivatives mean, with the help of Fig.24, which shows what happens when $x \to x + \mathrm{d}x$ and $y \to y + \mathrm{d}y$.



A: $z = f(x, y)$
B: $z = f(x + \mathrm{d}x, y)$
C: $z = f(x + \mathrm{d}x, y + \mathrm{d}y)$
D: $z = f(x, y + \mathrm{d}y)$

Figure 24

Example.

Let's find some second derivatives for a function of the third degree.

$$z = f(x, y) = x^3 + 4x^2y - 6xy^2 - 2y^3.$$

Differentiating, we get first derivatives

$$(\partial f/\partial x) = 3x^2 + 8xy - 6y^2,\ (\partial f/\partial y) = 4x^2 - 12xy - 6y^2,$$

and differentiating again we find four second derivatives:

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial x^2}\right) &= 6x + 8y, & \left(\frac{\partial^2 f}{\partial x \partial y}\right) &= 8x - 12y, \\ \left(\frac{\partial^2 f}{\partial y \partial x}\right) &= 8x - 12y, & \left(\frac{\partial^2 f}{\partial y^2}\right) &= -12x - 12y. \end{aligned}$$

Note that in the 'mixed' derivatives, where you differentiated once with respect to x and once with respect to y, the results came out to be exactly the same: it didn't matter which one you did first. This is true for *all* the functions we have called "well behaved" (meaning there is a unique value of the dependent variable z for every choice of the independent variables, the function is *continuous* and *smooth*, and so on).

Figure 24 shows four neighbouring points A,B,C,D on the surface: they are vertically above the corners of the rectangle at the bottom of the picture, which lies in the horizontal xy-plane. The broken lines indicate a plane passing through A, parallel to the base, so you can see the slopes of the edges of the sloping face. Thus, the slopes of AB and AD are

$$\begin{aligned} \text{slope}_{\text{AB}} &= \left(\frac{\partial f}{\partial x}\right) = \mathsf{D}_x f = f_x(x, y), \\ \text{slope}_{\text{AD}} &= \left(\frac{\partial f}{\partial y}\right) = \mathsf{D}_y f = f_y(x, y). \end{aligned}$$

Here the forms on the right show also the variables for Point A where the derivatives are evaluated. The slope of side BC, at Point B will be

$$\text{slope}_{\text{BC}} = \left(\frac{\partial f}{\partial y}\right) = \mathsf{D}_y\, f = f_y(x + \mathrm{d}x, y).$$

Notice that this is the slope in the y direction, where only x is changing, but it is evaluated using the variables for Point B. Why do we give no formula for the slope in the x direction? Simply because only y is changing as we go along BC and the 'x-slope' is thefore zero. Now look at points C and D and write down the slopes of the edges BC and CD. (Make a big drawing of Fig.24, so you can write on it and see what you're doing!)

In summary, starting from the point A, at height $z = f(x, y)$, you can increase x a little bit keeping y fixed, which gets you to B where $z = f(x + \mathrm{d}x, y)$. Then you can increase y by a small amount $\mathrm{d}y$, which gets you to C where $z = f(x + \mathrm{d}x, y{+}\mathrm{d}y)$; and then to D, whose height is $z = f(x, y{+}\mathrm{d}y)$. If you know the slopes in the x- and y-directions *at the starting point* A, you can estimate the heights of the other points B,C,D without re-calculating the function: for example, the height of B will be $z + \mathrm{d}z$ with $\mathrm{d}z \approx (\partial f/\partial x)\mathrm{d}x$, where the approximation is good if $\mathrm{d}x$ is very small. This means you are taking the edge AB as a straight line, touching the surface at Point A, when in fact it may be slightly curved. (Look back at Fig.14 in Chapter 2.) The same is true for point D, using the slope of AD, which is in the y-direction. But to get $z + \mathrm{d}z$ at point C you have to start at B; and you don't know the slope of the edge BC, though it looks as if it will be roughly the same as that of AD. So we must allow for *two* changes – the change in the function value f and the change in its first derivative f'_y as you go from A to B. To estimate the change in f_y as x changes from x at A to $x + \mathrm{d}x$ at B, we need to introduce a *second* derivative. The y-slope f_y is also a function of both x and y, so we can say

$$\mathsf{D}_x\, f_y(x, y) = \mathsf{D}_x \mathsf{D}_y\; f(x, y) = f_{xy}(x, y). \qquad (6.7)$$

In the same way, the x-slope $f_x(x, y)$ of AB will change at the rate

$$\mathsf{D}_y\, f_y(x, y) = \mathsf{D}_y \mathsf{D}_x\; f(x, y) = f_{yx}(x, y). \qquad (6.8)$$

It follows that

$$\mathrm{slope}_{\mathrm{BC}} \approx \mathrm{slope}_{\mathrm{AD}} + f_{xy}(x, y) \times \mathrm{d}x$$

and, on multiplying by dy, that the increase in z on going from B to C will be

$$f_y(x, y)dy + f_{xy}(x, y)\mathrm{d}x\mathrm{d}y.$$

It's now clear that, to allow for the change of slope as we go from one edge of the piece of surface ABCD to the opposite edge (e.g. from AC to BD), we have to know the 'mixed' second derivatives f_{xy} and f_{yx}. And to allow for changing slope f_x as we go along either edge (e.g. from A to B) we also have to know the second derivatives f_{xx} and f_{yy}.

Now that we have a picture of what we're doing, we can use the mathematics of Section 5.3. But if you find this a bit difficult you can skip the small print and go straight to equation (6.12).

Taylor's theorem in the form (5.17) gives us a way of expanding any function $f(x+h)$ around the general point x, in powers of h. If we use our new notation for the derivatives it takes the form

$$f(x+h) = f(x)+hf_x(x)+(h^2/2!)f_{xx}(x)+(h^3/3!)f_{xxx}(x)+. \, , \tag{6.9}$$

where h is the change we're making in x and all the derivatives are evaluated at the starting point $h=0$. For a function of more than one variable, there's not much to change: $f(x,y)$ now gives the height z at Point A in Fig.24 and if we keep y fixed, moving along the surface ABCD only in the x-direction, (6.9) can be re-written as

$$\begin{aligned} f(x+h,y) &= f(x,y)+hf_x(x,y)+(h^2/2!)f_{xx}(x,y) \\ &\quad + (h^3/3!)f_{xxx}(x,y) + \; ... \, , \end{aligned} \tag{6.10}$$

where all the derivatives are *partial* derivatives, with y treated as a constant.

Now let's think about changing the second variable y, starting from A and letting $y \rightarrow y+k$ while the first variable is kept fixed, with its original value. The function $f(x,y)$ and all its derivatives – also functions of x, y – can then be expanded in powers of k, using y in place of x and k in place of h. So moving across the surface ABCD in the y-direction, for any fixed value of x, we can say

$$\begin{aligned} f(x, y+k) &= f(x,y) + kf_y(x,y) + (k^2/2!)f_{yy}(x,y).. , \\ f_x(x, y+k) &= f_x(x,y) + kf_{yx}(x,y)... , \\ f_{xx}(x, y+k) &= f_{xx}(x,y)... . \end{aligned} \tag{6.11}$$

Notice that, when there are two subscripts on a function, the first one refers to the latest variable to be changed.

Finally, we can use the results in (6.11) to expand $f(x + h, y + k)$ in powers of both h and k. Changing y to $y + k$ in (6.10), the first three terms give (up to second order in the small quantities h, k)

$$f(x+h, y+k) = \\ f(x, y+k) + hf_x(x, y+k) + (h^2/2!)f_{xx}(x, y+k) + \; ... \, .$$

And now we can substitute for the three functions on the right, which are given in (6.11). The result is

$$\begin{aligned} &f(x+h, y+k) = \\ &\quad f(x,y) + kf_y(x,y) + (k^2/2!)f_{yy}(x,y) + \; ... \\ &\quad + h \times [f_x(x,y) + kf_{yx}(x,y) + \; ... \quad \text{(Term 2)}, \\ &\quad + (h^2/2!) \times [f_{xx}(x,y) + \; ... \quad \text{(Term 3)}, \end{aligned}$$

To make this look pretty, and easy to remember, we can re-arrange it:

$$\begin{aligned} &f(x+h, y+k) = f(x,y) \\ &\quad + hf_x(x,y) + kf_y(x,y) \\ &\quad + \tfrac{1}{2}[h^2 f_{xx}(x,y) + 2hkf_{yx}(x,y) + k^2 f_{yy}(x,y)]. \end{aligned} \tag{6.12}$$

Some consequences

What comes out from what we have done? The first consequence is one that follows from (6.12): it is that for any well-behaved function of two independent variables the order in which we do the partial differentiations doesn't matter. In symbols,

$$\mathsf{D}_x\mathsf{D}_y f(x,y) = \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left(\frac{\partial^2 f}{\partial y \partial x}\right) = \mathsf{D}_y\mathsf{D}_x f(x,y), \tag{6.13}$$

where we show both notations for the second derivatives. In other words the operators D_x and D_y **commute**: $\mathsf{D}_x\mathsf{D}_y = \mathsf{D}_y\mathsf{D}_x$.

It came out like that in the **Example** following equation (6.6), but we didn't try to *prove* it. Now we can, because by changing the order in which we let $x \to x+h$ and $y \to y+k$, we'd arrive at

$$\begin{aligned} f(x+h, y+k) &= f(x,y) \\ &\quad + hf_x(x,y) + kf_y(x,y) \\ &\quad + \tfrac{1}{2}[h^2 f_{xx}(x,y) + 2hkf_{xy}(x,y) + k^2 f_{yy}(x,y)], \end{aligned}$$

instead of (6.12). The alternative results must be identical, giving the value of z at the same point $(x+h, y+k)$; but the two expressions for $f(x+h, y+k)$ differ only in the mixed second derivative, which is $f_{yx}(x,y)$ in (6.12) but $f_{xy}(x,y)$ in the new expression. This proves the equality in (6.13).

A second consequence concerns the *existence* of a well-behaved function $z = f(x,y)$, which can be described by means of a surface on which every pair of values x, y defines a unique point at height z. For such a surface we have noted that, to first order in the infinitesimal changes $\mathrm{d}x(=h), \mathrm{d}y(=k)$,

$$\mathrm{d}z = \left(\frac{\partial f}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)\mathrm{d}y. \tag{6.14}$$

The sum of the two terms on the right is called the **total differential** of $z = f(x, y)$. But suppose we are told that the change in some quantity is related to $\mathrm{d}x, \mathrm{d}y$ by

$$\Delta = M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y,$$

where only the coefficients of $\mathrm{d}x$ and $\mathrm{d}y$ are given as functions of x and y. How can we tell whether Δ is the total differential of some unique well-behaved function of the variables x, y?

There is a simple test; because if $\Delta = \mathrm{d}z$ it must be given by (6.14) and therefore

$$M(x, y) = \left(\frac{\partial f}{\partial x}\right), \quad N(x, y) = \left(\frac{\partial f}{\partial y}\right).$$

This requires, in view of (6.13), that

$$\left(\frac{\partial M}{\partial y}\right) = \left(\frac{\partial N}{\partial x}\right), \tag{6.15}$$

each side of the equation being equal to the mixed second derivative

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right).$$

When these conditions are satisfied, $M\mathrm{d}x + N\mathrm{d}y$ is said to be an **exact differenial** and can be equated to the total differential (6.14) of some well behaved function $z = f(x, y)$.

What all this means, in terms of the hill-climbing picture Fig.23, is that you can go from point A, at height $z = f(x, y)$, to any other point on the surface and it doesn't matter what route you take. You could go a short distance North, then a long way East, and so on. But it is the total distance gone in each direction that determines the height you reach at the end: z is a *unique* and *single-valued* function of x and y. This would not be the case if there was a cliff between two possible routes: you could arrive at two different points, with exactly the same values of x and y, one at the top of the cliff and the other at the bottom! In that case z would *not* be single-valued, smooth and continuous.

In Physics and Chemistry there are hundreds of examples in which some measurable quantity depends on two or more others $(x, y, \ ... \ )$ and the total change in the quantity depends on the route you take in changing the independent variables. To make life easier we usually try to find quantities that depend only on the final values of the variables. These 'path-independent' functions are specially important. To find them we'll need the results of this Section.

6.2 Differential equations

We've already met several kinds of **differential equation**, in which there is a relationship between a function $y = f(x)$ and its derivatives $f'(x) = \mathrm{d}y/\mathrm{d}x, f''(x) = \mathrm{d}^2y/\mathrm{d}x^2$, etc.; and to *solve* the equation you have to find the form of the function that satisfies the given relationship. There's not enough room in one chapter of a short book to say much about differential equations so we'll have to be content with a few simple examples.

First-order equations

In Chapter 2, the velocity v of a freely falling body was a solution of the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a \quad (a = \text{acceleration due to gravity}).$$

Here the acceleration a is a constant, usually denoted by g. It is the rate of increase of v with time t, and the solution of the equation is $v = v_0 + at$, where v_0 is another constant. You can check this by doing the differentiation, finding $\mathrm{d}v/\mathrm{d}t = a$. And you can see what the constant v_0 means by putting $t = 0$ in your solution, which then gives $v = v_0$ at time $t = 0$ (i.e. the time at which you let go, taken as 'zero').

The differential equation here is a **first-order** equation because it contains only a *first derivative*; and the **general solution** contains *one* constant (v_0). A *particular* solution follows if you choose this constant so as to satisfy a **boundary condition**, which is here $v = v_0$ – at the 'boundary' of the time variable ($t = 0$ at the start of the motion).

Another first order equation we've met is the one that defines the exponential function. If $y = e^x$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x = y.$$

This function describes the growth of a population (Section 1.4 of Chapter 1) when the variables x and y are replaced by n (number of generations, which measures the time) and N (number of people at time t). In equation (1.9) the solution is given as

$$N = N_0 \exp(cn)$$

and if you differentiate this with respect to n you find

$$\frac{\mathrm{d}N}{\mathrm{d}n} = cN_0 \exp(cn) = cN.$$

This seems a bit more difficult. $\mathrm{d}N/\mathrm{d}n = cN$ doesn't give the derivative in terms of the independent variable n, so with the usual notation it would read $\mathrm{d}y/\mathrm{d}x = cy$. But it's still a first-order equation and the general solution contains one constant, N_0, which you need to fix by using the boundary condition: at the start, taking $n = 0$, the number of people will be $N = N_0 e^0 = N_0$, so you've fixed the constant to get the correct particular solution. [I know you need at least 2, but out of millions that's something you can forget about!] The other constant c is something you are given, like the g in the first example, it's part of the problem – not the solution.

Second-order equations

A **second-order** differential equation is one that contains up to second derivatives. Its general form is

$$\frac{d^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = r(x), \tag{6.16}$$

where $p(x), q(x), r(x)$ are given functions of the independent variable x and $y = f(x)$ is the general solution required. If the function $r(x)$ on the right is zero the equation is called **homogeneous**, but if it is non-zero the equation becomes **inhomogeneous** and the solutions are of a different kind. In this introduction we'll keep to the simplest type where $r(x) = 0$ and the coefficients $p(x), q(x)$, and $r(x)$ are *constants*. We'll be looking at examples of the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + by = c, \tag{6.17}$$

which is a **linear equation with constant coefficients**. The simplest possible equation of this kind is $\mathrm{d}^2y/\mathrm{d}x^2 =$ constant, which we met in equation (2.2) and put in words at the beginning of this Section: the velocity of a freely falling body increases at a constant rate. We'll take this as our first Example, putting in a bit more detail.

Example 1. Free fall – and parachutes

In calculus language, $\mathrm{d}v/\mathrm{d}t = g$ (the acceleration due to gravity); and since $v = \mathrm{d}s/\mathrm{d}t$, where s is the distance fallen at time t, the basic equation becomes

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = g. \tag{6.18}$$

In equation (2.4) of Chapter 2, we found a solution, $s = f(t)$, by a graphical method: on putting it in our present notation it reads $s = \frac{1}{2}gt^2$. To make sure this function does satisfy (6.18) we just differentiate it twice, getting first $\mathrm{d}s/\mathrm{d}t = \frac{1}{2}g \times 2t = gt$, and then $\mathrm{d}^2s/\mathrm{d}t^2 = g$. But this can't be the *general* solution, because a second-degree equation should have a solution with two arbitrary constants – and here we don't even have one.

To get the general solution we only need remember that every time we differentiate we lose any constant term; so when we differentiated $\mathrm{d}s/\mathrm{d}t$ ($= v$) we could have added a constant (call it v_0 as it has the dimensions of velocity – 'distance over time') and the constant would disappear in the differentiation. We could replace v by $v + v_0$ and the result would still satisfy the same differential equation. The same is true for the distance fallen, s, if we change s to $s+s_0$. So the *general* solution seems to be

$$s = s_0 + v_0t + \tfrac{1}{2}gt^2 \qquad (6.19)$$

and it's easy to check that this also satisfies (6.18). The constants clearly correspond to particular *boundary conditions* at $t = 0$. If we take $t = 0$ as the instant when we let the body fall, then $s = s_0$ is the distance at the start of the fall (wherever we measure it from). And since $v = \mathrm{d}s/\mathrm{d}t = v_0 + gt$ it is clear that $v = v_0$ when $t = 0$ – it is the 'initial' velocity. If we let the object fall from *rest*, at the position $s_0 = 0$, then $s = \frac{1}{2}gt^2$ is the *particular* solution of (6.18) corresponding to these boundary conditions.

But is (6.18) really correct? – because the so lution tells us that the speed of any freely falling body will go on increasing forever, becoming as big as you please if you wait long enough. That isn't what really happens, so the equation we've used can't be exactly right. In fact it's only an approximation, because it leaves out anything that *opposes* gravity by acting so as to *reduce* the downward speed. Instead of (6.18) we should really be using

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = g - kv, \tag{6.20}$$

where the constant k depends on what the body is *falling through*. The new term, $-kv$, is proportional to the velocity (it is doubled if you double v), and it has a negative sign because it acts *against* gravity, trying to *reduce* the rate of fall. For a body falling through the air, k is so small it is usually neglected. But it is still there, as you'll know if you've ever jumped into a river from a high bridge: as you go down faster and faster you can feel the air rushing past as it tries to slow you down – but k is not big enough and you hit the water with a great splash! If instead you jump from a 'plane', or a helicopter, you'd better take a parachute with you: it fills with air, like an umbrella in the wind, and slows your fall until you hit the ground at only a low speed. It gives you a much bigger value of k. And now you can see why that is important and why we want to solve (6.20).

To get a solution let's go first for the velocity, turning the equation round so it becomes

$$\frac{1}{g - kv}\frac{\mathrm{d}v}{\mathrm{d}t} = 1. \tag{6.21}$$

If you're very smart you will be able to see that this connects with what you learnt in Section 4.3: you can *integrate* the equation, with respect to time,by writing

$$\int \frac{1}{g - kv}\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t = t, \qquad (6.22)$$

and then recognising the left-hand side as

$$\int \frac{\mathrm{d}v}{g - kv} = -\frac{1}{k}\log(g - kv).$$

(Remember the standard integral given in (4.24) and put $((g - kv) = u)$ Another way of solving (6.21) is to turn it round by using (4.24): in that case it becomes dt/dv = 1/(g – kv) and integration gives

$$t = \int \frac{\mathrm{d}v}{(g - kv)} = \int \frac{1}{u}\frac{\mathrm{d}v}{\mathrm{d}u}\mathrm{d}u = -\frac{1}{k}\log u$$

– exactly as before.

On putting this result in (6.21) we get $t = -(1/k)\log u$, or (since integration always brings in an arbitrary constant, C say)

$$t = -(1/k)\log(g - kv) + C.$$

The constant is fixed by the boundary conditions: at $t = 0$ we suppose the body falls from rest ($v = v_0 = 0$). So $C = (1/k)\log g$; and at any later time t it follows that

$$kt = \log g - \log(g - kv) = \log\left(\frac{g}{g - kv}\right),$$

since $\log A - \log B = \log(A/B)$.

The final result looks clearer in exponential form (check it!):

$$(k/g)v = 1 - e^{-kt}. \tag{6.23}$$

This satisfies the boundary conditions, for when $t = 0$ it gives $v = v_0 = 0$ (starting fom rest), and it predicts a **terminal velocity**

$$v_T = g/k, \tag{6.24}$$

when t becomes indefinitely large. This is what you need to know if you're designing parachutes: how big must the constant k be to guarantee a safe landing? You can also calculate how far you will fall in reaching any given speed, by integrating the velocity, v ($= \mathrm{d}s/\mathrm{d}t$), given in (6.23). Think about it!

Example 2. The simple pendulum

Near the beginning of Book 1 we talked about measuring things, in particular **time**. You've all seen pendulum clocks, where a heavy body on a string, or stick, swings from side to side. Each double-swing of the 'bob' (forward and then backward) marks out a 'unit of time' and if we want to know how long something takes we just count the *number of swings* between starting and finishing: that's what the clock does in giving you t as a number of units.

The swinging pendulum is an **oscillator** and the time taken for one complete oscillation is called its **period**. To make it swing you have to displace the bob, by pulling it a distance, y say, from its **equilibrium position** where it's at rest:

then you let go and the pendulum oscillates. The motion is described by a second-order differential equation of the form (note that the independent variable is t and that the displacement y depends on t):

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\omega^2 y, \tag{6.25}$$

where ω ('omega' is the letter we often use in talking about oscillations) is a *constant* and the minus sign means that the side-to-side velocity of the bob (namely $\mathrm{d}y/\mathrm{d}t$) is always directed towards the equilibrium position where $y = 0$. We'll be looking for the general solution of this equation, but first look at the results earlier in this chapter to see if we have any functions that might give us 'ready-made' solutions.

Table 1 in Section 4.1 collects some key results. There you find that if $y = \sin x$ then $\mathrm{d}y/\mathrm{d}x = \cos x$; and if $y = \cos x$ then $\mathrm{d}y/\mathrm{d}x = -\sin x$. These results are easily extended to $y = \sin ax$, $y = \cos ax$. Thus, changing the variable by putting $ax = u$ and using the rule (2.22), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin ax = \frac{\mathrm{d}}{\mathrm{d}u}\sin u \times \frac{\mathrm{d}u}{\mathrm{d}x} = \cos u \times a = a\cos ax,$$

with a similar result when $y = \cos ax$. Together,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin ax = a\cos ax, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos ax = -a\sin ax. \tag{6.26}$$

Now do the two differentiations one after the other, to get

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin ax = \frac{\mathrm{d}}{\mathrm{d}x}(a\cos ax) = a(-a\sin ax).$$

That means that if you differentiate the function $y = \sin ax$ twice with respect to the variable x you get back the *original function*, multiplied by $-a^2$! And it's the same for the function $y = \cos ax$:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \sin ax = -a^2 \sin ax, \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2} \cos ax = -a^2 \cos ax.$$

In the differential equation we want to solve, namely (6.25), the independent variable was called t (not x) and the displacement y was a function of t. With this notation, the last two equations become

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sin at = -a^2 \sin at, \\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \cos at = -a^2 \cos at \end{aligned} \tag{6.27}$$

and give us two solutions of (6.25)! If we put $a = \omega$ the solutions are thus

$$y_1 = \sin \omega t, \qquad y_2 = \cos \omega t. \tag{6.28}$$

But we were looking for a *general* solution, with two arbitrary constants – where are they? It's clear that either solution in (6.28) can be multiplied by a constant, A or B say, and will still be a solution. To see this it's enough to write the original equation in the form

$$\mathsf{L}y = \mathsf{D}^2 y + \omega^2 y = 0, \tag{6.29}$$

where $\mathsf{L} = \mathsf{D}^2 + \omega^2$ is a **linear operator**, and if y is any solution then so is Ay. Still more generally, if y_1 and y_2 are any two solutions then so is $y = Ay_1 + By_2$; for

$$\mathsf{L}y = \mathsf{L}(Ay_1 + By_2) = A(\mathsf{L}y_1) + B(\mathsf{L}y_2) = A \times 0 + B \times 0.$$

And now we have the general solution:

$$y = A \sin \omega t + B \cos \omega t. \tag{6.30}$$

Remember that the sine and cosine functions, defined by series in (1.6), have the form shown in Fig.6: when the argument of the function, ωt, increases by 2π the value of the function starts repeating, the **cycle** of all distinct values has been completed. The **period** T is thus defined by $\omega T = 2\pi$, while the **frequency** 'nu' of the oscillation – the number of complete oscillations per unit time – is $\nu = 1/T$: thus

$$T = \frac{2\pi}{\omega}, \qquad \nu = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{6.31}$$

Example 3. Making music

Many musical instruments depend on the **vibrations** of a tightly stretched string, which is either 'plucked' (by pulling it to one side and then letting go) or 'rubbed' (by stroking it with a 'bow'). The shape of the vibrating string may be, for example, as in Fig.25:

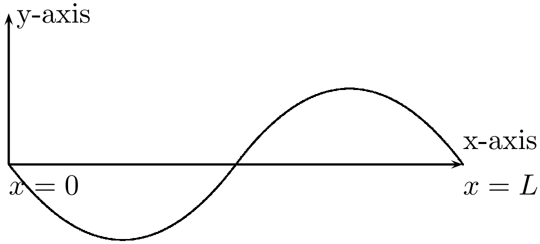


Figure 25

Let's think about a string of length L, stretched between its two ends: if you pluck it and wait for it to 'settle down' it will vibrate and make a musical sound.

Here the ends of the string are fixed at $x = 0$ and $x = L$ and the displacement $y = f(x)$ is greatly magnified. But what about the time t? In fact we're not going to have an ordinary differential equation this time, but a *partial* differential equation; because the displacement y at any point x on the string will depend also on t as the string vibrates up and down. We'll need an equation to determine $y = f(x, t)$, which is a function of *two* variables.

The equation you need looks very simple. The displacement y, at any point x, varies with time according to (using ∂ instead of d as there are *two* variables, x, y, as well as t)

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}. \qquad (6.32)$$

Here c^2 is a positive constant (that's why we've written it as a square), which depends on how *tightly* you stretch the string and on how *heavy* it is. The tighter the string, the faster it vibrates; but the heavier it is, the slower it moves. These things will become clear in Book 4, but for now we only want to get a solution to (6.32) which is a "second-order partial differential equation in two variables".

Fig.25 suggests that, even when it moves, the string may have a certain 'shape' $y = y(x)$ at any instant of time. Can we find *particular* solutions, in which the string keeps its shape but simply moves up and down? – the displacement at any point getting bigger and smaller as t changes. To see if this is possible let's try for a solution of the form

$$y(x,t) = F(x)T(t). \tag{6.33}$$

This is an example of 'separation of the variables', which is often useful in solving partial differential equations even when it doesn't usually lead to a *general* solution.

Now put (6.33) in (6.32) and see what happens. We'll then have to find the two functions, $F(x)$ and $T(t)$, such that

$$\left(\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right)T(t) = \frac{1}{c^2}F(x)\left(\frac{\mathrm{d}^2T}{\mathrm{d}t^2}\right), \tag{6.34}$$

where we've written *ordinary* derivatives instead of partial derivatives when we're differentiating functions of only *one* variable, like $F(x)$ and $T(t)$.

This doesn/t look any simpler. But now divide both sides of the equation by $F(x)T(t)$ and you get

$$\frac{1}{F}\left(\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right) = \frac{1}{c^2T}\left(\frac{\mathrm{d}^2T}{\mathrm{d}t^2}\right), \tag{6.35}$$

where everything on the left side depends only on x and everything on the right depends only on t – for all values of both variables! How can that be possible? If the two sides were equal at one moment we could come back a bit later and only the right-hand side would have changed – so how could the two sides still be equal?

Equation (6.35) could still hold good only if both sides were equal to the same thing, a **separation constant** C, not depending in any way on either x or t. So now we have "separated the variables" and have one equation to determine $F(x)$ and an independent equation to determine $T(t)$.

The equation for $T(t)$ is

$$\frac{1}{T}\left(\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}\right) = Cc^2,$$

where the separation constant appears on the right. This last equation has exactly the same form as (6.25) if we put $Cc^2 = -\omega^2$; so we already know the solution for the time-factor T: it is given in (6.30) as

$$T(t) = A\sin\omega t + B\cos\omega t. \qquad (6.36)$$

We needn't keep both terms, however, because they're really the same function, wiggling up and down as you change t by moving along the time axis. One wiggle is shown in Fig.25, starting at $y = 0$ when $t = 0$ (plotting t-values horizontally, in place of x-values, and if we measure time from that point then the sine term is enough: the cosine term looks the same, but pushed on along the t-axis so that it gives $y = B$ at $t = 0$ – which we don't want.

The expression on the left in (6.34), put equal to the same separation constant C, gives the equation to determine $F(x)$: it is

$$\frac{1}{F}\left(\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right) = C = -(\omega/c)^2$$

and again the solution will in general be a combination of the two terms $\sin(\omega/c)x$ and $\cos(\omega/c)x$. In this case also, the general solution is not required, because $y = 0$ at the two ends of the string where $x = 0$ and $x = L$: the cosine term doesn't fit, since $\cos 0 = 1$, so the solution must be of the form $F(x) = \sin(\omega/c)x$. And this must be zero when $x = L$. These *boundary conditions* are very important: the sine function can be zero only for values of its argument ($(\omega/c)x$) which are integer multiples of π. So the only acceptable functions $F(x)$ must have $\omega L/c = n\pi$ where n is an integer. Thus the function $F(x)$, which determines the 'shape' of the vibrating string, must be of the form $F(x) = A\sin\left(\frac{n\pi}{L}\right)x$. And if we attach the time factor, the first term in (6.36), then the particular solution we need will be

$$y(x,t) = A\sin(\omega/c)x\sin\omega t = A\sin\left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}\right)ct. \tag{6.37}$$

Here A is an arbitrary constant called the **amplitude** of the vibration: and since the sine function takes only values lying between $+1$ and -1 the displacement of any point on the string must always be between $\pm A$.

This solution tells us all we need to know about the vibrating string. The period and frequency of the vibration follow as in (6.31), on substituting $\omega = (n\pi c/L)$ it follows that

$$T = \frac{2L}{nc}, \qquad \nu = \frac{nc}{2L}. \tag{6.38}$$

Notice especially that there is one solution for every integer value $n = 1, 2, 3, ...$ and that these alternative solutions arise as a result of the *boundary conditions*, $y = 0$ at the two ends of the string ($x = 0, x = L$) where it is fixed. The vibration indicated in Fig.25 corresponds to $n = 2$ and in general n indicates the number of 'half-waves' that can be fitted onto the string. The 'mode of vibration' for any given value of n is called the "nth normal mode". The one for $n = 1$ is called the "fundamental", while the ones for higher values of n are the "overtones". If you pluck a string near the middle, the musical sound you get corresponds mainly to the fundamental, while if you pluck it away from the middle you'll get a mixture of overtones. The frequency (or 'pitch') of the sound can also be changed by stretching the string tighter or using a heavier one, thus changing the value of c. All these things are important in the design of musical instruments.

6.3 Eigenfunctions – and how we can use them

Example 3 in the last Section brought in a very important new idea, opening up a whole new field of mathematics. A differential equation which contains a 'parameter' (like the separation constant C or the related constant ω), that has solutions *only for certain special values of the parameter*, is called an **eigenvalue equation**. The 'allowed' values of the parameter are called **eigenvalues** and the corresponding solutions, like (6.37) for example, are the **eigenfunctions**.

In earlier Sections we've used two methods of 'fitting' a given function, $y = F(x)$ say, by supposing it can be represented approximately by a power series

$$F(x) \approx = f(x) = a_0 + a_1 x + a_2 x^2 + \ ... \qquad (6.39)$$

and choosing the numerical values of the coefficients by some suitable rule. This worked well in Section 4.5, where we took just three points and chose the coefficients in the polynomial $y = A + Bx + cx^2$ so that it would reproduce the 'true' values of the function $y = 1/(1 + x^2)$ at those points – failing a little bit at other points. In the second method, we used Taylor's theorem to get the coefficients in (6.39) in terms of the *derivatives*, $(\mathrm{dy}/\mathrm{dx}), (\mathrm{d^2y}/\mathrm{dx}^2)$, etc of the given function, all evaluated at a *single reference point* $x = 0$. Thus, we found $a_0 = y_0 = F(0),\ a_1 = (\mathrm{d}y/\mathrm{d}x)_0 = F'(0),\ a_2 = \frac{1}{2}(\mathrm{d}^2y/\mathrm{d}x^2)_0 = \frac{1}{2}F''(0)$, and so on. This method gives *all* the coefficients in the power series $f(x)$ and can represent $F(x)$ *exactly* if we take an infinite number of terms. But we have to be able to get all the derivatives of $F(x)$ and calculate their values for $x = 0$ – which can be a lot of work! If you stop at, say, three terms, you'll find that $f(x)$ can be a good approximation for small values of x but gets worse and worse as x increases.

Now we're going to use a third method, which doesn't focus on a finite number of points or on one point only. Instead we try to represent the given function as a linear combination of *eigenfunctions* (all defined of course in the range of x values in which we are interested) coming from some suitable differential equation. Let's take as an example the function $F(x)$ shown in Fig.26 by the heavy line – a nasty-looking function with a constant value $y = 1$ for x in the interval (0,0.5), but then dropping vertically to $y = 0$ and staying there until x reaches the upper boundary $x = 1$. It's a function defined in the interval (0,1), with a discontinuity at $x = 0.5$, and we'll try to represent it as a combination of the *eigenfunctions* $\phi_n(x) = A\sin(n\pi x)$, $n = 1, 2, 3,$ ... which determine the normal modes of a vibrating string of length $L = 1$. This looks impossible – but we'll try!

If we take only the first N eigenfunctions, we'll get an N-term approximation to $F(x)$:

$$F_N(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ ... \ + c_N\phi_N(x) \qquad (6.40)$$

and our job will be to choose the coefficients so as to get a *best* approximation to $F(x)$.

Generally, the curves we get on plotting $F(x)$ and $F_N(x)$ will differ, as indicated in Fig.26 for the 1-term approximation where we've taken $c_1 = 1$. The first approximation is then $F_1(x) = \phi_1(x)$. It differs from $F(x)$ by $\Delta(x) = F(x) - F_N(x)$ at all points in the range, sometimes by a positive amount (shown by the up-arrow) and sometimes by a negative amount (down-arrow). So it's no good adding these differences for all points on the curve (which will mean integrating $\Delta(x)$) to get a measure of how poor the approximation is; for cancellations could lead to zero even when the curves were very different. It's really the *magnitude* of $\Delta(x)$ that matters, or its *square* – which is always positive.

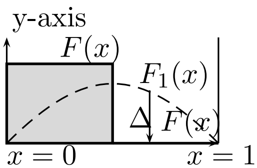


Figure 26

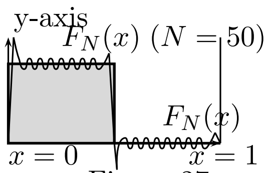


Figure 27

So instead let's measure the difference by $|F(x) - F_N(x)|^2$, at any point, and the 'total difference' by

$$D = \int \Delta(x)^2 dx = \int |F(x) - F_N(x)|^2 dx. \qquad (6.41)$$

The integral gives the sum of the areas of all the strips of height Δ^2 and width dx. This quantity will measure the *error* when the whole curve is approximated by $F_N(x)$ and we'll only get a really good fit, over the whole range of x, when D is close to zero.

The coefficients c_n should be chosen to give D its lowest possible value and we already know how to do that: in Chapter 2, Exercise 9, for a function of one variable, we found a minimum value by first looking for a 'turning point' where $(\mathrm{d}f/\mathrm{d}x) = 0$; and then checked that it really *was* a minimum, by verifying that $(\mathrm{d}^2 f/\mathrm{d}x^2)$ was positive. It's just the same here, except that we look at the variables *one at* *a time, keeping the others constant.* Remember too that it's the coefficients c_n that we're going to vary, not x.

Now when you put (6.40) into (6.41) it looks like you'll get a big mess: but if you keep your head and write it all down there's nothing that you can't do!

You first get (dropping the usual variable x where it's not needed)

$$D = \int |F - F_N|^2 \mathrm{d}x = \int F^2 \mathrm{d}x + \int F_N^2 \mathrm{d}x - 2\int F F_N \mathrm{d}x. \tag{6.42}$$

So there are three terms to differentiate – only the last two really, because the first doesn't contain any c_n and so will disappear when you start differentiating. These two terms are very easy to deal with if we make use of the special properties of eigenfunctions. You don't need to prove these: we'll just write them down for the ones we are using.

The eigenfunctions for the vibrating string are solutions of the differential equation (6.35), which is

$$\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = -\omega^2 F.$$

They have the simple form $F(x) = A\sin(n\pi x)$ for a string of unit length ($L = 1$) and it is convenient to choose the constant A as $\sqrt{2}$, as we'll see in a moment. The first few eigenfunctions are thus

$$\begin{aligned} \phi_1(x) &= \sqrt{2}\sin(\pi x), \quad \phi_2(x) = \sqrt{2}\sin(2\pi x), \\ \phi_3(x) &= \sqrt{2}\sin(3\pi x), \quad \phi_4(x) = \sqrt{2}\sin(4\pi x), \ \ldots \end{aligned} \tag{6.43}$$

and we're using them for x in the interval (0,1). The special properties we need are very simple:

$$\int_0^1 \phi_n(x)^2 \mathrm{d}x = 1, \qquad \int_0^1 \phi_m(x)\phi_n(x)\mathrm{d}x = 0 \quad \text{for m} \neq \text{n}. \tag{6.44}$$

Every function is said to be **normalized to unity** (as in the first equation); and any two *different* functions are said to be **orthogonal** (the second equation). The factor $\sqrt{2}$ in (6.43) was chosen to normalize the functions. (Check that it does so by evaluating the integral!)

Knowing these two properties, we can go back to (6.42) and differentiate the last two terms, with respect to each c_n (one at a time, holding the others fixed: the first of the tw2o terms leads to

$$\frac{\partial}{\partial c_n}\int_0^1 F_N^2 \mathrm{d}x = \frac{\partial}{\partial c_n} c_n^2 \int_0^1 \phi_n(x)^2 \mathrm{d}x; = 2c_n$$

while the second one gives

$$-2\frac{\partial}{\partial c_n}\int_0^1 F F_N \mathrm{d}x =$$
$$-2\frac{\partial}{\partial c_n} c_n \int_0^1 F(x)\phi_n(x)\mathrm{d}x = -2c_n\langle F|\phi_n\rangle,$$

where a special notation has been used for the integral $\int_0^1 F(x)\phi_n(x)\mathrm{d}x$, which is called the **scalar product** of the two functions $F(x)$ and $\phi_n(x)$. Thus

$$\langle F|\phi_n\rangle = \int_0^1 F(x)\phi_n(x)\mathrm{d}x. \tag{6.45}$$

We can now do the differentiation of the difference function D in (6.42). The result is

$$\frac{\partial D}{\partial c_n} = -2c_n - 2c_n\langle F|\phi_n\rangle$$

and this tells us immediately how to choose the coefficients in the N-term approximation (6.39) so as to get the best possible fit to the given function $F(x)$:

$$c_n = \langle F|\phi_n\rangle \quad \text{(for all } n\text{)}. \tag{6.46}$$

So it's really very simple: you just have to evaluate one integral to get any coefficient you want. And once you've got it, there's never any need to change it in getting a better approximation. You can make the expansion as long as you like by adding more terms, but the coefficients of the ones you've already done are *final*.

The one-term approximation $F(x) \approx F_1(x)$ is clearly very poor, as you can see from Fig.26. It gives a rounded peak in the middle of the range, instead of the square step on the left-hand side and y close to zero in the range $x = 0.5$ to $x = 1$. But in a two-term approximation $F_2(x) = c_1\phi_1(x) + c_2\phi_2(x)$ you will add on some of the function $\phi_2(x)$ shown in Fig.25: this will build up the function in the left-hand half of the interval (0,1) but reduce it in the right-hand half. If you go on adding more terms, always with the right coefficients, given in (6.46), you'll get closer and closer to the given function $F(x)$. Figure 27 shows (roughly) the kind of fit you get with fifty terms. Notice that there is still a small wiggle in the ranges where $F(x)$ is flat; and that where the slope is discontinuous, the function jumping vertically at $x = 0$ and $x = 0.5$, the difference $\Delta(x)$ is bigger, making a 'spike' in the approximation. But in the end, with very many terms, any spikes become so narrow as to be in many ways unimportant. This kind of approximation introduces the idea of **convergence in the mean**, the difference between the function and its 'approximant' tending to zero over most of the given range, but still allowing finite differences at certain *points*.

Eigenfunction expansions are very important in Physics and the **property of finality** holds for *all* such expansions, not only for those where the eigenfunctions describe the vibration of a string and satisfy the simple differential equation (6.28). The same approach holds for eigenfunctions in one or many variables and can be used as an easy and practical way of getting into quite difficult theory.

In this book you've taken great strides into mathematics; and that must be, for the moment, our final step!

Exercises

1) Find the partial derivatives,

$$\frac{\partial f}{\partial x}, \qquad \frac{\partial f}{\partial y},$$

of the functions (a) $z = f(x, y) = x^3 + 4x^2y + 2xy^2 + 3y^3$
(b) $z = f(x, y) = \sqrt{x^2 + 3y^2}$

(*Hint*: Write this as $u^{1/2}$ where $u = x^2 + 3y^2$ and use the 'chain rule' (2.22) for both derivatives, holding the second variable constant.) (c) $z = f(x, y) = y \sin(x^2 + y^2)$
(*Hint*: This is of the form $y \times u(x, y)$: use the rule (2.21) for the product and (2.22) for the second factor, holding x or y constant.)

2) From the first derivatives found in Exercise 1(a), differentiate again to find the *second* partial derivatives

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y}, \quad \frac{\partial^2 z}{\partial y \partial x}.$$

Note the equality of the two 'mixed' second derivatives.

3) Do the same as in the last Exercise, but starting from the first derivatives of the function in (c) of Exercise 1, and confirm that the mixed second derivatives are again equal.

4) Look at the function $z = \sqrt{x^2 + y^2}$, which gives the height z of a point on a 2-dimensional surface, as a function of horizontal distances measured from the origin ($x = y = 0$) in the x- and y-directions. (These are the x- and y-*coordinates* of the point, as you'll remember from Book 2.)

Make a sketch to show the form of this particular surface, along with a 'contour map', in which each contour is labelled by its height z.

Get an expression for the total differential dz, in terms of small displacements dx, dy from the origin.

5) Use the first few terms of the Taylor expansion in (6.12) to find how the height of a point on the surface $z = f(x^2 + y^2)$

changes as you go from the point P with coordinates x, y to a point Q with coordinates $x + h, y + k$.

Make some numerical tests, starting from the point $x = 1, y = 2$ to see how accurately you can approximate $\Delta z = f(x + h, y + k) - f(x, y)$ (the real change of height) by the differential $\mathrm{d}z$, evaluated using for example $h = k = 0.1$.

6) Suppose you are told that, by taking infinitesimal steps $\mathrm{d}x, \mathrm{d}y$ in the x- and y-directions, the height z of a point on some surface will change by $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y$, with $M(x, y) = (x^2 - y^2), \quad N(x, y) = xy$. Is this possible, in general, for any 'well-behaved' surface?

(*Hint*: For this to be true, you could make any sequence of steps, using this recipe, and always reach the same end point at some height $z = f(x, y)$, depending only on the final values of x, y. But the test (6.15) will show you this is simply not possible.)

7) Repeat Exercise 6, but assuming that $M(x, y) = 2x + y, \quad N(x, y) = x + 2y$. You will find that the test is satisfied. Find the equation of the surface, $z = f(x, y)$ and say what this result means.

8) In the text we found a solution of the differential equation for free fall of an object, taking account of the *resistence* of whatever it was falling through (e.g. air, water). The result (6.23) contains a constant k: what are its physical dimensions?

Try putting in some numerical values of k (in what units?), to get corresponding values of the terminal velocity (6.24). Choose a value and then calculate the velocity reached after

1,2, and 5 seconds of free fall. Suppose the fall starts 100 m above the ground. Integrate the velocity equation to find the distance fallen at time t. What will your speed be when you hit the ground?

(*Hint*: If you've forgotten about 'dimensions' look back Section 2.1, where you'll also find the approximate value of the 'acceleration (a) due to gravity' – usually denoted by g.)

9) Work through Section 6.3 carefully to make sure you understand every step. Then show that, when the expansion coefficients c_n in (6.40) are determined using (6.45), the 'difference function' D in (6.42) takes the form $D = 1 - \sum_1^N c_n^2$. Can you say what this result means? Calculate the values of D_1 and D_2 for the 1- and 2-term approximations to the given function (shown shaded in Figs. 26 and 27).

(*Hint*: Do you remember that the sum of squares of the components of a vector (Section 6.1 of Book 2) gives the square of its length? Here the sum as $N \to \infty$ tends to 1: but what about the sum for only 1, 2,... or 50 terms?)

10) Now try to find a similar approximation to the function $F(x) = x$, which starts at $F(0) = 0$ and rises to $F(1) = 1$. Sketch the results for 1-, 2- and 3-term approximations and get corresponding values of D, which indicates the error in each approximation.

(*Hint*: Turn back to Chapter 4 for the integrals you'll need.)

Looking back –

In Book 3, you've learnt how mathematics can be used in describing **relationships** between the quantities you may want to measure – how the distance you travel (s, say) may depend on the time you take (t, say) and how you can put this in mathematical language by writing $s = f(t)$. You say this in words as "s is a **function** of t" where the f is just a name for the rule which tells how to get the value of s for any given value of t.

- Chapter 1. There are three main ways of describing a relationship $y = f(x)$: (i) by making a **table** showing values you choose for the **independent variable** x and the corresponding values of the **dependent variable** y; (ii) by 'plotting' the pairs of values $(x_1, y_1), (x_2, y_2)$, ... to get a **graph** of the relationship; or (iii) by using 'standard' mathematical functions, such as $y = x^n$, $y = \sin x$, $y = e^x$, and so on. Here you've learnt about the simplest standard functions and know what their graphs look like.

- In Chapter 2 you've met all the main ideas of the **calculus**: **differentiating** to find how fast $y = f(x)$ changes when you change x; and **integrating** to find the area under the curve $y = f(x)$ between limits at x_1 and x_2; and you know what all this means and how it can be used. The result of differentiating $y = f(x)$, written $dy/dx = f'(x)$, is the **derivative**, of the function and is a new function of x.

- In Chapter 3, you found the derivatives of a number of standard functions and were then able to differentiate *anything* that you could express in terms of them, as sums or products and so on.

- Chapter 4 looked at the problem of *integrating* any given function, which is more difficult because there's no simple rule and you have to look for special 'tricks'. But integration is so useful that you need to be able to do it, even when you can't find the right tricks. In that case, we found *numerical* methods, which require only a table of values of x, y and give you what you need, using simple arithmetic.

- In Chapter 5, we came back to **power series**. in which a given function is represented in the form $y = f(x) = a_0 + a_1x + a_2x^2 + \ ...$. **Taylor's theorem** shows us how to choose the coefficients.

- Finally, in Chapter 6, we took a first look at some things not usually done before university. You won't need them yet, but when you do they'll seem like 'old friends' – no harder than what you've done already.

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